QUANTUM DIVIDED POWER ALGEBRA, Q-DERIVATIVES AND SOME NEW QUANTUM GROUPS

Naihong Hu

Abstract. The discussions in the present paper arise from exploring intrinsically the structure nature of the quantum n-space. A kind of braided category $\mathcal{GB}$ of Λ-graded θ-commutative associative algebras over a field k is established. The quantum divided power algebra over k related to the quantum n-space is introduced and described as a braided Hopf algebra in $\mathcal{GB}$ (in terms of its 2-cocycle structure), over which the so called special q-derivatives are defined so that several new interesting quantum groups, especially, the quantized polynomial algebra in n variables (as the quantized universal enveloping algebra of the abelian Lie algebra of dimension n), and the quantum group associated to the quantum n-space, are derived from our approach independently of using the R-matrix. As a verification of its validity of our discussion, the quantum divided power algebra is equipped with a structure of $U_q(\mathfrak{sl}_n)$-module algebra via a certain q-differential operators realization. Particularly, one of the four kinds of roots vectors of $U_q(\mathfrak{sl}_n)$ in the sense of Lusztig can be specified precisely under the realization.

1. Introduction and Preliminaries

1.1 It is a known fact that associated to a simple Lie group G there exist two new Hopf algebra structures, namely, quantum group $k_q[G]$, and quantized universal enveloping algebra $U_q(\mathfrak{g})$. In particular, to $G = GL(n)$ or $SL(n)$ there correspond two (right) comodule-algebras (cf. [13]): the first one

$$k[A_q^{n|0}] = k\{x_1, \cdots, x_n\}/(x_jx_i - q\,x_ix_j, \quad i < j)$$

is the *quantum n-space* (i.e. *quantum symmetric algebra*) whereas the second one

$$k[A_q^{0|n}] = k\{x_1 \cdots, x_n\}/(x_i^2,\ x_j\,x_i + q^{-1}\,x_i\,x_j, \quad i < j)$$

is the *quantum exterior algebra* $\Lambda_q(n)$.

Supported in part by the National Natural Science Foundation of China (Grant No. 19731004), the Science Foundation of the University Doctoral Program CNCE, the Ministry of Education of China and the Shanghai Scientific and Technical Commission, as well as the Institute of Mathematic of ECNU.

Key words and phrases. quantum n-space, bicharacter, (braided) Hopf algebra, quantum divided power (restricted) algebra, q-derivatives, (Hopf) module algebra, quantum roots vectors.

Similar to the classical case, there exists a Hopf duality between $k_q[G]$ and $U_q(\mathfrak{g})$ (cf. [2], [6], [7], etc.). Hence, $k[A_q^{n|0}]$ and $k[A_q^{0|n}]$ should be (left) $U_q(\mathfrak{g})$-module algebras (cf. [1, 16]), where $\mathfrak{g} = \mathfrak{gl}_n$, $\mathfrak{sl}_n$. Thus, a natural problem arises from here, i.e. how to concretely realize the quantized universal enveloping algebra $U_q(\mathfrak{g})$ as certain q-differential operators over the associated quantum n-space such that $k[A_q^{n|0}]$ becomes a $U_q(\mathfrak{g})$-module algebra. According to [12], this question has its (non-commutative) geometric meaning.

1.2 For the case when $\mathfrak{g}$ is $\mathfrak{sl}_2$, this was solved independently by J. Wess and B. Zumino ([17]), S. Montgomery and S.P. Smith (in fact, their realization is just relative to the Woronowicz's Hopf subalgebra of $U_q(\mathfrak{sl}_2)$) ([12]), and C. Kassel ([7]). Kassel's treatment depends on complicated commutative operations concerning left- and right- multiplications so that it seems impossible to generalize to the general case $\mathbf{A}_{n-1}$ by following his approach (Note that in his realization, the two q-derivatives $\partial_q/\partial x$, $\partial_q/\partial y$ are commutative). For $\mathfrak{sl}_n$, T. Hayashi ([5]) gave a realization of $U_q(\mathfrak{sl}_n)$ over a polynomial ring $S = k[X_1, \cdots, X_{n+1}]$ (also see Jantzen's book [6]), however, his realization cannot make the polynomial ring S a $U_q(\mathfrak{sl}_n)$-module algebra in the sense of [1, 16].

In the present paper, we first establish a kind of braided category $\mathcal{GB}$ of Λ-graded θ-commutative algebras over a field k in section 2, where Λ is a free abelian group (of finite rank) and θ is a bicharacter (or called a 2-cocycle) on Λ, and then describe $k[A_q^{n|0}]$ as a braided Hopf algebra in the category $\mathcal{GB}$, which is a generalization of super-Hopf algebra (cf. [15]) or a kind of braided Hopf algebra in the sense of S. Majid (cf. [10]), relative to a 2-cocycle defined on the $\mathbb{Z}^n$-graded structure of the quantum n-space. Instead of it, we introduce the notion of quantum divided power algebra we would work with. In section 3, we define a kind of q-differential operators (i.e. special q-derivatives) over it which differs from the above-mentioned and can be used to give the required realization of $U_q(\mathfrak{g})$ as in **1.1** (for $\mathfrak{g} = \mathfrak{gl}_n$, or $\mathfrak{sl}_n$). Also, we obtain several new quantum groups, for instance, the quantum group $\mathfrak{D}_q$ whose smash product relative to the quantum divided power algebra $\mathcal{A}_q$ yields a kind of quantum Weyl algebra (i.e. the algebra of quantum differential operators) distinguished from those appeared in the literature (for instance, [3], [5], [10], etc.) to the best of my knowledge. Of particular interest in our discussion, we are able to obtain the exact object of polynomial algebra in n variables in the context of quantum groups as well as the quantum group structure associated to the quantum n-space $k[A_q^{n|0}]$. On the other hand, with the realization in section 4, we consider the submodules structure of the quantum divided power (restricted) algebra, especially, we show that one of the four kinds of roots vectors of $U_q(\mathfrak{sl}_n)$ introduced by G. Lusztig in [9] can be specified precisely with those q-differential operators we defined.

For the sake of latter discussion, we recall the following notions.

1.3 Recall that a Hopf algebra (H, Δ, ϵ, S) over k means H is an algebra, $\Delta : H \to H \otimes H$ (the comultiplication) and $\epsilon : H \to k$ (the counit) are algebra homomorphisms and $S : H \to H$ (the antipode) plays the role of inverse. Here $H \otimes H$ is of the tensor product algebra structure. Call an (associative) algebra A over k an *H-module*

algebra (cf. [1], [16], etc.) if A has an (left) H-module structure such that

$$\text{(i)} \qquad h\,1_A = \epsilon(h)\,1_A,$$

$$\text{(ii)} \qquad h(a\,b) = \sum (h_{(1)}\,a)\,(h_{(2)}\,b),$$

for $h \in H$, $a, b \in A$ with $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$. Here the second condition means that the multiplication is a homomorphism of H-modules.

Given two automorphisms σ and τ of an algebra A, a linear endomorphism δ of A is called a (σ, τ)-*derivation* if

$$\text{(iii)} \qquad \delta(a\,a') = \sigma(a)\,\delta(a') + \delta(a)\,\tau(a'),$$

for $a, a' \in A$ (cf. [7]).

A *quantum group* in the sense of Drinfeld (cf. [4]) is a non-commutative and non-cocommutative Hopf algebra. For super-quantum groups and super-Hopf algebras, however, the main difference is that the algebra structure on $H \otimes H$ uses the super-transposition (cf. [15]): $\psi(x \otimes y) = (\pm 1)^{|x||y|} y \otimes x$ (on homogeneous elements). More generally, by a *braided Hopf algebra* in a certain *braided category* means the algebra structure on $H \otimes H$ is provided by a certain "braiding" in the sense of Majid (cf. [10]). Precisely, between any two objects there is a tensor product that is commutative and associative up to isomorphism. The first of these isomorphisms is the braiding $\psi_{V,W} : V \otimes W \to W \otimes V$. For any objects U, V and W, the braiding obeys

$$\text{(iv)} \qquad \begin{aligned} \psi_{U\otimes V,W} &= (\psi_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes \psi_{V,W}), \\ \psi_{U,V\otimes W} &= (\mathrm{id}_V \otimes \psi_{U,W}) \circ (\psi_{U,V} \otimes \mathrm{id}_W). \end{aligned}$$

In addition, there should be an identity object $\underline{1}$ for $\otimes$, and one has $\psi_{\underline{1},V} = \mathrm{id} = \psi_{V,\underline{1}}$. In these formulae, the associativity isomorphism and isomorphisms such as $V \otimes \underline{1} \cong V \cong \underline{1} \otimes V$ are suppressed.

1.4 The *quantized universal enveloping algebra* $U_q(\mathfrak{sl}_n)$ (cf. [6], [7], [9], etc.) is the k-algebra generated by the symbols $\mathcal{K}_i^{\pm 1}$, e_i and f_i $(1 \le i \le n-1)$ with the following defining relations:

$$\text{(i)} \qquad \mathcal{K}_i\,\mathcal{K}_i^{-1} = \mathcal{K}_i^{-1}\,\mathcal{K}_i = 1, \qquad \mathcal{K}_i\,\mathcal{K}_j = \mathcal{K}_j\,\mathcal{K}_i,$$

$$\text{(ii)} \qquad \mathcal{K}_i\,e_j\,\mathcal{K}_i^{-1} = q^{a_{ij}}\,e_j, \qquad \mathcal{K}_i\,f_j\,\mathcal{K}_i^{-1} = q^{-a_{ij}}\,f_j,$$

$$\text{(iii)} \qquad [e_i, f_j] = \delta_{ij}\,\frac{\mathcal{K}_i - \mathcal{K}_i^{-1}}{q - q^{-1}},$$

$$\text{(iv)} \qquad \begin{aligned} & e_i^2\,e_j - (q + q^{-1})\,e_i\,e_j\,e_i + e_j\,e_i^2 = 0 \qquad (|i-j| = 1), \\ & e_i\,e_j = e_j\,e_i \qquad (|i-j| > 1), \end{aligned}$$

$$\text{(v)} \qquad \begin{aligned} & f_i^2\,f_j - (q + q^{-1})\,f_i\,f_j\,f_i + f_j\,f_i^2 = 0 \qquad (|i-j| = 1), \\ & f_i\,f_j = f_j\,f_i \qquad (|i-j| > 1), \end{aligned}$$

where $q \in k^*$ and (a_{ij}) is the Cartan matrix of type $\mathbf{A}_{n-1}$.

The Hopf algebra structure of $U_q(\mathfrak{sl}_n)$ is as follows:

$$
\begin{aligned}
&\text{(vi)} && \Delta(\mathcal{K}_i^{\pm 1}) = \mathcal{K}_i^{\pm 1} \otimes \mathcal{K}_i^{\pm 1},\\
&&& \epsilon(\mathcal{K}_i^{\pm 1}) = 1,\\
&&& S(\mathcal{K}_i^{\pm 1}) = \mathcal{K}_i^{\mp 1},\\
&\text{(vii)} && \Delta(e_i) = e_i \otimes \mathcal{K}_i + 1 \otimes e_i,\\
&&& \epsilon(e_i) = 0,\\
&&& S(e_i) = -e_i\, \mathcal{K}_i^{-1},\\
&\text{(viii)} && \Delta(f_i) = f_i \otimes 1 + \mathcal{K}_i^{-1} \otimes f_i,\\
&&& \epsilon(f_i) = 0,\\
&&& S(f_i) = -\mathcal{K}_i\, f_i.
\end{aligned}
$$

Let P be the weight lattice for $\mathfrak{gl}_n$. It is the free $\mathbb{Z}$-module of rank n with canonical basis $\{\varepsilon_i\}_{1\le i\le n}$, and $\{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \le i \le n-1\}$ is the set of simple roots of $\mathfrak{gl}_n$. $Q = \bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i \subset P$ is the root lattice of $\mathfrak{gl}_n$. Fix a symmetric bilinear form $\langle , \rangle : P \times P \longrightarrow \mathbb{Z}$ such that $\langle \varepsilon_i, \varepsilon_j\rangle = \delta_{ij}$ for $1 \le i, j \le n$. Through this pairing, we can identify P with its dual $P^* = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$.

Now we can state the presentation of $U_q(\mathfrak{gl}_n)$ as follows. Change (i), (ii) into

$$
\begin{aligned}
&\text{(ix)} && k_i\, k_i^{-1} = k_i^{-1}\, k_i = 1, \qquad k_i\, k_j = k_j\, k_i \qquad (1 \le i, j \le n),\\
&&& \mathcal{K}_i = k_i\, k_{i+1}^{-1} \qquad (1 \le i \le n-1),\\
&\text{(x)} && k_i\, e_j\, k_i^{-1} = q^{\langle \varepsilon_i, \alpha_j\rangle} e_j, \qquad k_i\, f_j\, k_i^{-1} = q^{-\langle \varepsilon_i, \alpha_j\rangle} f_j,
\end{aligned}
$$

but keep (iii)–(v) invariant. As for its Hopf algebra structure, we only need to replace $\mathcal{K}_i$ $(1 \le i \le n-1)$ in (vi) with k_i $(1 \le i \le n)$, except with the same items as (vii) and (viii).

1.5 As we know, the q-binomial coefficients are closely related to the study of quantizations $U_q(\mathfrak{g})$ of enveloping algebras. For our work here we need some known facts involving them.

Let $\mathbb{Z}[v, v^{-1}]$ be the Laurent polynmmial ring in variable v. For any integer $n \ge 0$ we define $[n]_v = \frac{v^n - v^{-n}}{v - v^{-1}} \in \mathbb{Z}[v, v^{-1}]$, and $[n]_v! = [n]_v [n-1]_v \cdots [1]_v$. It is well known that for two integers m, r with $r \ge 0$ we have (cf. [8]):

$$
\text{(i)} \qquad \begin{bmatrix} m \\ r \end{bmatrix}_v = \prod_{i=1}^{r} \frac{v^{m-i+1} - v^{-m+i-1}}{v^i - v^{-i}} \in \mathbb{Z}[v, v^{-1}].
$$

By (i), we get $\begin{bmatrix} m \\ r \end{bmatrix}_v = \frac{[m]_v!}{[r]_v!\,[m-r]_v!}$ if $0 \le r \le m$, $\begin{bmatrix} m \\ r \end{bmatrix}_v = 0$ if $0 \le m < r$, and $\begin{bmatrix} m \\ r \end{bmatrix}_v = (-1)^r \begin{bmatrix} -m+r-1 \\ r \end{bmatrix}_v$ if $m < 0$. We again set $\begin{bmatrix} m \\ r \end{bmatrix}_v = 0$ when $r < 0$.

Suppose now that k is a field and $q \in k^*$. By definition, we get $[n] := [n]_{v=q}$, $[n]! := [n]_{v=q}! \in k$ for $n \ge 0$ and $\begin{bmatrix} m \\ r \end{bmatrix} := \begin{bmatrix} m \\ r \end{bmatrix}_{v=q} \in k$ when v is specialized to be q. Note that the q-binomial coefficients $\begin{bmatrix} n \\ r \end{bmatrix}$ $(0 \le r \le n)$ can be defined recursively by

$$
\begin{bmatrix} n \\ r \end{bmatrix} = q^{r-n} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} + q^r \begin{bmatrix} n-1 \\ r \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
$$

The combinatorial formula below involving the q-binomial coefficients $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]$ is well known (cf. [8]).

$$\text{(ii)} \qquad \prod_{i=0}^{n-1}(1+q^{2i}\,x)=\sum_{r=0}^{n}q^{(n-1)r}\begin{bmatrix} n \\ r \end{bmatrix}x^r.$$

The situation when q is a primitive root of unity is of particular interest in quantum phenomenon. We here introduce a notion, so called the *characteristic* of q, which is defined as the minimal positive integer l such that $[\,l\,]=0$ and denoted by $\mathbf{char}(q)=l$. Also we define $\mathbf{char}(q)=0$ when q is generic (in this case, $[\,n\,]\neq 0$ for any non-zero $n\in\mathbb{Z}$). If $q\neq\pm1$, then $\mathbf{char}(q)=l\,(>2)$ implies two cases that q is a primitive $2l$-th root of 1, or a primitive l-th root of 1 but l odd. And vice versa.

The following Lemma is proved by Lusztig under the assumption $l\geq 3$ is odd.

Lemma. *Assume that $q\in k^*$ and $\mathbf{char}(q)=l\geq 3$.*

(1) *Let $m=m_0+m_1l$, $r=r_0+r_1l$ with $0\leq m_0$, $r_0<l$, m_1, $r_1\geq 0$ and $m\geq r$. Then $\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]=\left[\begin{smallmatrix} m_0 \\ r_0 \end{smallmatrix}\right]\binom{m_1}{r_1}$, where $\binom{m_1}{r_1}$ is an ordinary binomial coefficient.*

(2) *Let $m=m_0+m_1l$, $0\leq m_0<l$, $m_1\in\mathbb{Z}$. Then $\left[\begin{smallmatrix} m \\ l \end{smallmatrix}\right]=m_1$ if $m_1\geq 0$; and $\left[\begin{smallmatrix} m \\ l \end{smallmatrix}\right]=-(-1)^l m_1$ if $m_1<0$ (i.e. $m<0$).*

(3) *If $m=m_0+m_1l$, $m'=m_0'+m_1'l\in\mathbb{Z}$ with $0\leq m_0$, $m_0'<l$ satisfy $q^m=q^{m'}$, $\left[\begin{smallmatrix} m \\ l \end{smallmatrix}\right]=\left[\begin{smallmatrix} m' \\ l \end{smallmatrix}\right]$, then $m=m'$ if l is odd or l even but $m_1m_1'\geq 0$; and $m'=\overline{m}=m_0-m_1l$ if l is even but $m_1m_1'<0$.*

Proof. By using formula (ii) and following the same argument as the proofs of Proposition 3.2 & Corollary 3.3 in [8], we readily show that (1), 2) and the first claim of (3) hold. For the case where l is even (in this case, q must be a primitive $2l$-th root of 1) and $m_1m_1'<0$, we deduce from $q^m=q^{m'}$ that $m_0=m_0'$ and $m_1=m_1'+2r$. Since $m_1m_1'<0$, we can let $m_1>0$ and $m_1'<0$. By (2), we get $m_1=\left[\begin{smallmatrix} m \\ l \end{smallmatrix}\right]=\left[\begin{smallmatrix} m' \\ l \end{smallmatrix}\right]=-m_1'$, as required. □

2. Braided Hopf Algebra and Quantum Divided Power Algebra

2.1 Let $\alpha=(\alpha_1,\cdots,\alpha_n),\beta=(\beta_1,\cdots,\beta_n)\in\mathbb{Z}^n$ be any two integers n-tuples, and define a product of them by

$$\text{(i)} \qquad \alpha*\beta=\sum_{j=1}^{n-1}\sum_{i>j}\alpha_i\beta_j,$$

then from (i), one has

Lemma. *The product $*$ satisfies the following distributive laws:*

(1) $(\alpha+\beta)*\gamma=\alpha*\gamma+\beta*\gamma$,
$\alpha*(\beta+\gamma)=\alpha*\beta+\alpha*\gamma$, *in particular,*

(2) $\varepsilon_i*\beta=\sum_{s<i}\beta_s \quad (1\leq i\leq n)$,
$(\varepsilon_i-\varepsilon_{i+1})*\beta=-\beta_i \quad (1\leq i<n)$.

(3) $\beta * \varepsilon_i = \sum_{s>i} \beta_s \quad (1 \le i \le n)$,
$\beta * (\varepsilon_i - \varepsilon_{i+1}) = \beta_{i+1} \quad (1 \le i < n)$.

Here $\varepsilon_i = (\delta_{1i}, \cdots, \delta_{ni})$ $(1 \le i \le n)$ *is a basis of* $\mathbb{Z}^n$ *as* $\mathbb{Z}$*-module.* $\square$

Now for $\alpha, \beta \in \mathbb{Z}^n$ and $q \in k^*$, we define a mapping $\theta : \mathbb{Z}^n \times \mathbb{Z}^n \to k^*$ by

$$\theta(\alpha, \beta) = q^{\alpha*\beta - \beta*\alpha}. \tag{ii}$$

In particular,

$$\theta(\varepsilon_i, \varepsilon_j) = \begin{cases} q, & i > j, \\ 1, & i = j, \\ q^{-1}, & i < j. \end{cases} \tag{iii}$$

Obviously, the mapping θ has the following properties:

$$\theta(\alpha + \beta, \gamma) = \theta(\alpha, \gamma)\theta(\beta, \gamma), \tag{iv}$$
$$\theta(\alpha, \beta + \gamma) = \theta(\alpha, \beta)\theta(\alpha, \gamma), \tag{v}$$
$$\theta(\alpha, 0) = 1 = \theta(0, \alpha), \tag{vi}$$
$$\theta(\alpha, \beta)\theta(\beta, \alpha) = 1 = \theta(\alpha, \alpha). \tag{vii}$$

Actually, such a mapping θ with the above properties is a *bicharacter* of the additive group $\mathbb{Z}^n$.

Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}_+^n$ be any nonnegative-integers n-tuple, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be any nonzero monomial in $k[A_q^{n|0}]$, then $\{x^\alpha \mid \alpha \in \mathbb{Z}_+^n\}$ constitutes a canonical basis of $k[A_q^{n|0}]$. Thus by definition (see **1.1**), $k[A_q^{n|0}] = \bigoplus_{\alpha \in \mathbb{Z}^n} kx^\alpha$ is a $\mathbb{Z}^n$-graded algebra (with $x^\alpha = 0$ for $\alpha \notin \mathbb{Z}_+^n$). Set $x^{\varepsilon_i} = x_i$.

Remark. We will point out that the quantum n-space $k[A_q^{n|0}]$ has a so-called "braided" Hopf algebraic structure (with respect to the above defined bicharacter θ). Actually, we can show a more general fact in the next subsection.

2.2 Let Λ be a (finitely generated) abelian group and θ a bicharacter of the abelian group Λ (namely, θ is of properties **2.1** (iv) — (vii)). Recall a 2-*cocycle* η defined on Λ with coefficients in k^*, *i.e.*, a mapping $\eta : \Lambda \times \Lambda \to k^*$ satisfying the relation

$$\eta(\alpha, \beta)\, \eta(\alpha + \beta, \gamma) = \eta(\beta, \gamma)\, \eta(\alpha, \beta + \gamma). \tag{i}$$

Observe that any bicharacter θ of Λ is a 2-cocycle on Λ, which is useful to our next consideration.

Let $\mathcal{GA}$ denote a category of graded associative unitary algebras over k. That is, for any $(A, \Lambda_A) \in Ob(\mathcal{GA})$, $A = \bigoplus_{\alpha \in \Lambda_A} A_\alpha$ is a Λ_A-graded associative algebra with $k \subseteq A_0$ and $A_\alpha \cdot A_\beta \subseteq A_{\alpha+\beta}$, where Λ_A is an abelian group. Moreover, $\phi_{A,B} := (\phi, \varphi) : (A, \Lambda_A) \to (B, \Lambda_B)$ is a morphism between (A, Λ_A) and (B, Λ_B), if $\phi : A \to B$ is a graded algebra homomorphism and $\varphi : \Lambda_A \to \Lambda_B$ is a group homomorphism such that $\phi(A_\alpha) \subseteq B_{\varphi(\alpha)}$.

Now for any object $(A,\Lambda)\in Ob\,(\mathcal{GA})$, to θ an arbitrary bicharacter of Λ, we can associate an opposite object $(A^{\rm op},\Lambda)\in Ob\,(\mathcal{GA})$ as follows:

Denote $A^{\rm op}:=(A,\circ)$, where $A^{\rm op}=\bigoplus_{\alpha\in\Lambda}A_\alpha$. Define

$$\text{(ii)}\qquad a\circ b:=\theta(\alpha,\beta)\,ba,\qquad \forall\ a\in A_\alpha,\ b\in A_\beta.$$

Clearly, $A_\alpha\circ A_\beta\subseteq A_{\alpha+\beta}$, for any $\alpha,\ \beta\in\Lambda$. On the other hand, by the 2-cocycle property (i) of θ as a bicharacter, for any $a\in A_\alpha$, $b\in A_\beta$ and $c\in A_\gamma$, we have

$$\begin{aligned}(a\circ b)\circ c&=\theta(\alpha,\beta)\,\theta(\alpha+\beta,\gamma)\,c(ba)\\&=\theta(\beta,\gamma)\,\theta(\alpha,\beta+\gamma)\,(cb)a=a\circ(b\circ c).\end{aligned}$$

This means $A^{\rm op}$ is an associative algebra. Thus $(A^{\rm op},\Lambda)\in Ob\,(\mathcal{GA})$.

Consider a subcategory $\mathcal{GC}$ in category $\mathcal{GA}$, whose objects consist of Λ-graded θ-commutative algebras over k (where Λ is an arbitrary abelian group, θ is an arbitrary bicharacter of Λ), *i.e.*, any object $(A,\Lambda_A;\theta_A)$ in $\mathcal{GC}$ is called Λ_A-*graded* θ_A-*commutative*, if $(A,\Lambda_A)\in Ob\,(\mathcal{GA})$ and $x\cdot y=\theta_A(\alpha,\beta)\,y\cdot x$, $\forall\ x\in A_\alpha,\ y\in A_\beta$. Moreover, $\Phi_{A,B}:=(\phi_{A,B},\tilde\varphi):(A,\Lambda_A;\theta_A)\to(B,\Lambda_B;\theta_B)$ is a morphism, if $\phi_{A,B}:(A,\Lambda_A)\to(B,\Lambda_B)$ is a morphism in category $\mathcal{GA}$ such that $\tilde\varphi(\theta_A)=\theta_B$, *i.e.*, $\theta_A(\alpha,\beta)=\theta_B(\varphi(\alpha),\varphi(\beta))$.

For any $(A,\Lambda,\theta)\in Ob\,(\mathcal{GC})$, by (ii), we notice that $a\circ b=\theta(\alpha,\beta)\,ba=ab$, for any $a\in A_\alpha$, $b\in A_\beta$. This means the opposite of any object in the category $\mathcal{GC}$ coincides with the itself, which is the case we have in the category of (usual) commutative algebras.

Examples. Any commutative algebra over k in the usual sense can be considered an object in $\mathcal{GC}$ with a trivial grading relative to a trivial group. The polynomial algebra $k[t_1,\cdots,t_n]$ is a $\mathbb{Z}^n$-graded θ-commutative algebra with a trivial bicharacter θ (means $\theta(\alpha,\beta)\equiv1$ for any $\alpha,\ \beta\in\mathbb{Z}^n$). Again, the quantum n-space $k[A_q^{n|0}]=\bigoplus_{\alpha\in\mathbb{Z}^n}kx^\alpha$ is a $\mathbb{Z}^n$-graded θ-commutative algebra with the bicharacter θ of $\mathbb{Z}^n$ defined in **2.1** (ii).

Now assume that Λ is a free abelian group and $\theta:\Lambda\times\Lambda\to k^*$ is a non-trivial bicharacter of Λ. Suppose that $F=\bigoplus_{\alpha\in\Lambda}F_\alpha\in Ob\,(\mathcal{GA})$ is a free object, that is, F is a free Λ-graded associative algebra (with 1).

Let $I=\langle\,x\cdot y-\theta(\alpha,\beta)\,y\cdot x,\ \forall\ x\in F_\alpha,\ y\in F_\beta,\ \forall\ \alpha,\ \beta\in\Lambda\,\rangle$ denote the Λ-graded ideal generated by all homogeneous elements of form $x\cdot y-\theta(\alpha,\beta)\,y\cdot x$, $x\in F_\alpha$, $y\in F_\beta$. Set $\mathcal{F}:=F/I$. Thus $\mathcal{F}$ is a free Λ-graded θ-commutative associative algebra over k, that is, $\mathcal{F}\in Ob\,(\mathcal{GC})$.

Remark. In such Λ-graded θ-commutative associative algebras, "θ-commutative" is well-defined due to the properties **2.1** (iv)—(vii) of the bicharacter θ, on the other hand, the 2-cocycle property (i) of the θ ensures the compatibility between "θ-commutativity" and "associativity".

2.3 Fix an abelian group Λ and a bicharacter θ of Λ, we consider a subcategory $\mathcal{GB}$ in category $\mathcal{GC}$ relative to the pair (Λ,θ), where $\forall\ (A,\Lambda;\theta),\ (B,\Lambda;\theta)\in Ob\,(\mathcal{GC})$, the morphisms between $(A,\Lambda;\theta)$ and $(B,\Lambda;\theta)$ in $\mathcal{GC}$ are of forms: $\Phi_{A,B}=(\phi_{A,B},\mathrm{id})=$

$((\phi, \mathrm{id}), \mathrm{id}) \equiv \phi : (A, \Lambda; \theta) \to (B, \Lambda; \theta)$, where $\phi : A \to B$ is a graded algebra homomorphism such that $\phi(A_\alpha) \subseteq B_\alpha$ for all $\alpha \in \Lambda$.

Note that $\mathcal{GB}$ is a *braided category* in the sense of section **1.3**. First of all, there exists an identity object $\underline{1} = k = \bigoplus_{\alpha\in\Lambda} k_\alpha \in Ob(\mathcal{GB})$ with $k_\alpha = \delta_{0,\alpha}\, k$. Next, for any $U = \bigoplus_{\alpha\in\Lambda} U_\alpha$, $V = \bigoplus_{\beta\in\Lambda} V_\beta \in Ob(\mathcal{GB})$, we have

$$U \otimes V = \bigoplus_{\gamma} (U\otimes V)_\gamma = \bigoplus_{\gamma=\alpha+\beta} U_\alpha \otimes V_\beta \in Ob(\mathcal{GB}).$$

Define the mapping $\psi_{U,V} : U \otimes V \to V \otimes U$ as

(i) $$\psi(x \otimes y) = \theta(\alpha, \beta)\, y \otimes x, \quad \text{for } x \in U_\alpha,\ y \in V_\beta.$$

The properties of a bicharacter (see **2.1** (iv)—(vii)) ensure that the mapping ψ is a braiding in the sense of **1.3** (iv). In particular, for any algebra $H \in \mathcal{GB}$, we have its opposite object, $H^{\mathrm{op}} \equiv H \in Ob(\mathcal{GB})$, and its tensor object, $H \otimes H \in Ob(\mathcal{GB})$, whose algebra structure is given by

(ii) $$(a\otimes b)(c\otimes d) = a\psi(b\otimes c)d = \theta(\alpha,\beta)\, ac\otimes bd, \quad \text{for } a, d \in H, b \in H_\alpha, c \in H_\beta.$$

Let $u(H)$ denote the group of invertible elements of H.

Theorem. *Let Λ be a free abelian group with a basis $\{\alpha_1, \cdots, \alpha_n\}$, θ a bicharacter of Λ and $\mathcal{GB}$ the braided category relative to the pair (Λ, θ). Suppose that $H = \bigoplus_{\alpha\in\Lambda} H_\alpha \in Ob(\mathcal{GB})$ such that $u(H) = k^*$. If H has no zero divisors $\neq 0$ and $\{\, a_{ij_i} \in H_{\alpha_i} \mid 1 \le j_i \le s_i, 1 \le i \le n \,\}$ is a set of generators for H. Then (H, Δ, ϵ, S) is a braided-commutative Hopf algebra relative to a braiding $\psi : H\otimes H \to H \otimes H$ defined by $\psi(a\otimes b) = \theta(\alpha, \beta)\, b \otimes a$ for $a \in H_\alpha$, $b \in H_\beta$, where the mappings Δ, ϵ and S defined below*

$$\begin{aligned} \Delta : H \to H \otimes H, &\qquad \Delta(a) = a\otimes 1 + 1 \otimes a, \quad \textit{for } a \in H_{\alpha_i}, \\ \epsilon : H \to k, &\qquad \epsilon(a) = \delta_{0,\alpha_i} a, \quad \textit{for } a \in H_{\alpha_i}, \\ S : H \to H^{\mathrm{op}}, &\qquad S(a) = -a, \quad \textit{for } a \in H_{\alpha_i} \end{aligned}$$

are the morphisms in the braided category $\mathcal{GB}$ (where $S(ab) = \theta(\alpha,\beta)\, S(b)S(a) = S(a)\circ S(b)$, $\forall\ a \in H_\alpha$, $b \in H_\beta$).

In particular, any free object $\mathcal{F}$ in $\mathcal{GB}$ constructed in **2.2** *is a braided-commutative Hopf algebra.*

Proof. First we need to check that Δ, ϵ and S preserve the algebraic relations of H.

For $a \in H_{\alpha_i}$, $b \in H_{\alpha_j}$, using (ii), we have

$$\begin{aligned} \Delta(a)\Delta(b) &= (a\otimes 1 + 1 \otimes a)(b \otimes 1 + 1\otimes b) \\ &= ab\otimes 1 + \theta(\alpha_i, \alpha_j) b \otimes a + a \otimes b + 1 \otimes ab \\ &= \theta(\alpha_i,\alpha_j)(ba\otimes 1 + b\otimes a + \theta(\alpha_j, \alpha_i) a \otimes b + 1 \otimes ba) \\ &= \theta(\alpha_i, \alpha_j)\Delta(b)\Delta(a), \\ \epsilon(ab) &= \delta_{0,\alpha_i}\delta_{0,\alpha_j} ab = 0 = \theta(\alpha_i,\alpha_j)\epsilon(ba), \\ S(ab) &= \theta(\alpha_i, \alpha_j)\, ba = ab = S(\theta(\alpha_i, \alpha_j)\, ba). \end{aligned}$$

Based on the consideration above, as well as the actions of Δ and ϵ on the generators of H, it is readily to see that $(1\otimes\Delta)\Delta=(\Delta\otimes 1)\Delta$ and $(1\otimes\epsilon)\Delta=1=(\epsilon\otimes 1)\Delta$ hold. Also, since $S(a)+S(1)a=0$, $\forall\, a\in H_{\alpha_i}$, we get $m\circ(S\otimes 1)\circ\Delta=\eta\circ\epsilon=m\circ(1\otimes S)\circ\Delta$ (where (H,m,η) is the algebra structure of H). Hence, $(H,m,\eta,\Delta,\epsilon,S)$ is a braided-commutative Hopf algebra. $\square$

Corollary. *The quantum n-space $k[A_q^{n|0}]$ with respect to the bicharacter θ of $\mathbb{Z}^n$ given in* **2.1** (ii) *is a $\mathbb{Z}^n$-graded braided-commutative Hopf algebra with $\Delta(x_i)=x_i\otimes 1+1\otimes x_i$, $S(x_i)=-x_i$, and $\epsilon(x_i)=0$ for $x_i\in k[A_q^{n|0}]_{\varepsilon_i}$, and where the braiding $\psi(a\otimes b)=\theta(\alpha,\beta)\, b\otimes a$ for $a\in k[A_q^{n|0}]_\alpha$, $b\in k[A_q^{n|0}]_\beta$.* $\square$

2.4 Here we will equip the quantum n-space $k[A_q^{n|0}]$ with a divided power structure when $\mathbf{char}(q)=0$. More generally, we can introduce a quantum divided power algebra $\mathcal{A}_q(n)$ for an arbitrary $q\in k^*$ as follows.

Let $\mathcal{A}_q(n):=\langle x^{(\alpha)}\mid\alpha\in\mathbb{Z}_+^n\rangle$ be a vector space over k generated by the (divided power) basis $x^{(\alpha)}$ $(\alpha\in\mathbb{Z}_+^n)$ with $x^{(0)}=1$. Define the multiplication in $\mathcal{A}_q(n)$ by

$$\text{(i)}\qquad x^{(\alpha)}\,x^{(\beta)}=q^{\alpha*\beta}\begin{bmatrix}\alpha+\beta\\ \alpha\end{bmatrix}x^{(\alpha+\beta)}=\theta(\alpha,\beta)\,x^{(\beta)}\,x^{(\alpha)},$$

where $\left[\begin{smallmatrix}\alpha+\beta\\ \alpha\end{smallmatrix}\right]:=\prod_{i=1}^n\left[\begin{smallmatrix}\alpha_i+\beta_i\\ \alpha_i\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}\alpha_i+\beta_i\\ \alpha_i\end{smallmatrix}\right]=\frac{[\alpha_i+\beta_i]!}{[\alpha_i]![\beta_i]!}$ for $\alpha_i,\,\beta_i\in\mathbb{Z}_+$. By slight abuse of notation, we also write $x_i=x^{(\varepsilon_i)}$.

Obviously, $\mathcal{A}_q(n)$ with the above multiplication (i) forms an associative algebra, and we call it a *quantum divided power algebra*.

In particular, when $\mathbf{char}(q)=0$, for $x^\alpha\in k[A_q^{n|0}]$, we set $x^{(\alpha)}:=\frac{1}{[\alpha]!}x^\alpha$ for $\alpha\in\mathbb{Z}_+^n$, where $[\,\alpha\,]!:=\prod_{i=1}^n[\,\alpha_i\,]!$. According to the multiplication of $k[A_q^{n|0}]$, we see that the algebraic structure $k[A_q^{n|0}]$ coincides with $\mathcal{A}_q(n)$, and $\{x^{(\alpha)}\mid\alpha\in\mathbb{Z}_+^n\}$ forms a (divided power) basis of $k[A_q^{n|0}]$ with $x^{(0)}=1$, which gives the *divided power structure* over the quantum n-space $k[A_q^{n|0}]$. In the case, $\{x_i\mid 1\le i\le n\}$ is a set of generators of $\mathcal{A}_q$.

Of particular interest to us in introducing $\mathcal{A}_q(n)$ is the case when $\mathbf{char}(q)=l\ge 3$, in which we can obtain a finite dimensional quantum divided power algebra.

Set $\tau=(l-1,\cdots,l-1)$. By $\alpha\le\beta$ means $\alpha_i\le\beta_i$ for all i. Denote $\mathcal{A}_q(n,\mathbf{1}):=\langle x^{(\alpha)}\mid\alpha\in\mathbb{Z}_+^n,\alpha\le\tau\rangle$. Note that $[\,l\,]=0$. Especially for $0\le s,t<l$, we have $\left[\begin{smallmatrix}s+t\\ s\end{smallmatrix}\right]=0$ if $s+t\ge l$. Consequently, the subspace $\mathcal{A}_q(n,\mathbf{1})$ is closed under the multiplication (i) in $\mathcal{A}_q(n)$, namely, $\mathcal{A}_q(n,\mathbf{1})$ forms a divided power subalgebra of $\mathcal{A}_q(n)$, which is l^n-dimensional. In this case, $(x^{(\alpha)})^l=0$ for any $x^{(\alpha)}\in\mathcal{A}_q(n,\mathbf{1})$ with $\alpha\ne 0$ so we say $\mathcal{A}_q(n,\mathbf{1})$ a *quantum restricted divided power algebra.*

Proposition. *Suppose that k is a field of characteristic zero,* $\mathbf{char}(q)=l\ge 3$. *Then the quantum divided power algebra $\mathcal{A}_q$ is generated by elements x_i and $x_i^{(l)}$ $(1\le i\le n)$; and its quantum restricted divided power algebra $\mathcal{A}_q(n,\mathbf{1})$ is generated by x_i $(1\le i\le n)$. In addition, when l is odd, $x_i^{(l)}$ $(1\le i\le n)$ are central elements of $\mathcal{A}_q(n)$ and $\mathcal{A}_q(n)\cong\mathcal{A}_q(n,\mathbf{1})\otimes_k k[\,x_1^{(l)},\cdots,x_n^{(l)}\,]$ (as algebras).*

Proof. For any $m \in \mathbb{Z}_+$, let $m = m_0 + m_1 l$ such that $0 \le M_0 \le l-1$, $m_1 \ge 0$. By using Lemma 1.5, we get $\begin{bmatrix} m \\ m_0 \end{bmatrix} = 1$, and conclude from (i) that

$$\text{(ii)} \qquad \big(x_i^{(l)}\big)^{m_1} = \begin{bmatrix} 2\,l \\ l \end{bmatrix}\begin{bmatrix} 3\,l \\ l \end{bmatrix}\cdots\begin{bmatrix} m_1 l \\ l \end{bmatrix} x_i^{(m_1 l)} = m_1!\, x_i^{(m_1 l)},$$

$$x_i^{(m)} = x_i^{(m_0)}\, x_i^{(m_1 l)} = \frac{1}{m_1!}\, x_i^{(m_0)}\, \big(x_i^{(l)}\big)^{m_1}.$$

$$x^{(\alpha)}\, x_i^{(l)} = \theta(\alpha, l\varepsilon_i)\, x_i^{(l)}\, x^{(\alpha)} = x_i^{(l)}\, x^{(\alpha)} \qquad (\textit{for}\ \ q^l = 1).$$

Note that $\mathbf{char}(q) = l$ being odd means $q^l = 1$. These imply the statements. □

By analogy of Corollary 2.3, we have

Corollary. *The quantum divided power algebra $\mathcal{A}_q(n)$ in the case when $\mathbf{char}(q) = 0$ or when $\mathbf{char}(q) = char\,(k) = p$ (a prime), together with the quantum restricted divided power algebra $\mathcal{A}_q(n, \mathbf{1})$ in the case when $\mathbf{char}(q) = char\,(k) = p$, is a $\mathbb{Z}^n$-graded braided-commutative Hopf algebra relative to the bicharacter θ given in* **2.1** (ii). □

In the latter discussion, we shall prefer to work with the quantum divided power algebra rather than the quantum n-space.

3. q-Derivatives, Quantum Groups and Quantum Weyl Algebra

3.1 For simplicity of notation, we let $\mathcal{A}_q$ denote $\mathcal{A}_q(n)$ for any $\mathbf{char}(q)$, or $\mathcal{A}_q(n, \mathbf{1})$ only under the case $\mathbf{char}(q) = l\,(> 2)$. Consider the algebra automorphisms σ_i $(1 \le i \le n)$ of $\mathcal{A}_q$ defined by

$$\text{(i)} \qquad \sigma_i(x^{(\beta)}) = q^{\langle \beta,\, \varepsilon_i\rangle}\, x^{(\beta)} = q^{\beta_i}\, x^{(\beta)}, \qquad \forall\, X^{(\beta)} \in \mathcal{A}_q.$$

When $q = 1$, one has $\sigma_i = \mathrm{id}$. Obviously, $\sigma_i\sigma_j = \sigma_j\sigma_i$.

Define $\frac{\partial_q}{\partial\, x_i}$ as the special q-derivatives over $\mathcal{A}_q$ by

$$\text{(ii)} \qquad \frac{\partial_q}{\partial\, x_i}(x^{(\beta)}) = q^{-\varepsilon_i * \beta}\, x^{(\beta-\varepsilon_i)}, \qquad \forall\, x^{(\beta)} \in \mathcal{A}_q.$$

For convenience, we briefly let ∂_i denote $\frac{\partial_q}{\partial\, x_i}$.

For $\alpha \in \mathbb{Z}^n$, denote $\Theta(\alpha)$ by the algebra automorphisms of $\mathcal{A}_q$:

$$\text{(iii)} \qquad \Theta(\alpha)(x^{(\beta)}) = \theta(\alpha, \beta)\, x^{(\beta)}, \qquad \forall\, x^{(\beta)} \in \mathcal{A}Wq,$$

where θ is the bicharacter on $\mathbb{Z}^n$ defined in **2.1**. Thus we have

Proposition. (1) *$\Theta(\alpha)\Theta(\beta) = \Theta(\alpha+\beta)$, in particular, $\Theta(-\alpha_i) = \sigma_i\sigma_{i+1}$, for a simple root $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ in root system of type $\mathbf{A}_{n-1}$.*

(2) *∂_i is a $(\Theta(-\varepsilon_i)\sigma_i^{\pm 1}, \sigma_i^{\mp 1})$-derivation of $\mathcal{A}_q$, namely,*

$$\partial_i(x^{(\beta)}\, x^{(\gamma)}) = \partial_i(x^{(\beta)})\, \sigma_i^{\mp 1}(x^{(\gamma)}) + \big(\Theta(-\varepsilon_i)\sigma_i^{\pm 1}\big)(x^{(\beta)})\, \partial_i(x^{(\gamma)}).$$

(3) $\partial_i\,\partial_j = \theta(-\varepsilon_i, -\varepsilon_j)\,\partial_j\,\partial_i = \theta(\varepsilon_i, \varepsilon_j)\,\partial_j\,\partial_i$.

(4)
$$\begin{aligned} x^{(\alpha)}(x^{(\beta)}\,x^{(\gamma)}) &= (x^{(\alpha)}\,x^{(\beta)})\,x^{(\gamma)} \\ &= \theta(\alpha,\beta)\,x^{(\beta)}\,(x^{(\alpha)}\,x^{(\gamma)}) \\ &= \Theta(\alpha)(x^{(\beta)})\,(x^{(\alpha)}\,x^{(\gamma)}). \end{aligned}$$

(5) $\sigma_i(x^{(\beta)}\,x^{(\gamma)}) = \sigma_i(x^{(\beta)})\,\sigma_i(x^{(\gamma)})$,
$\Theta(\alpha)(x^{(\beta)}\,x^{(\gamma)}) = \Theta(\alpha)(x^{(\beta)})\,\Theta(\alpha)(x^{(\gamma)})$.

(6) $x^{(\alpha)}\partial_i$ *is a* $(\Theta(\alpha-\varepsilon_i)\sigma_i^{\pm1}, \sigma_i^{\mp1})$*-derivation of* $\mathcal{A}_q$*, namely,*
$$\big(x^{(\alpha)}\partial_i\big)(x^{(\beta)}\,x^{(\gamma)}) = \big(x^{(\alpha)}\partial_i\big)(x^{(\beta)})\,\sigma_i^{\mp1}(x^{(\gamma)}) + \big(\Theta(\alpha-\varepsilon_i)\sigma_i^{\pm1}\big)(x^{(\beta)})\,\big(x^{(\alpha)}\partial_i\big)(x^{(\gamma)}).$$

Proof. (1) The first claim is due to the property of bicharacter θ (see **2.1** (iv)). The second follows from $\Theta(-\alpha_i)(x^{(\gamma)}) = \theta(-\alpha_i,\gamma)x^{(\gamma)} = \sigma_i\sigma_{i+1}(x^{(\gamma)})$ (by **2.1** (ii) and Lemma 2.1 (2) & (3)).

(2) Noting that $[m+m'] = [m]\,q^{\mp m'} + [m']\,q^{\pm m}$, and

$$\text{(iv)} \qquad \begin{bmatrix}\beta+\gamma\\ \beta\end{bmatrix} = \begin{bmatrix}\beta+\gamma-\varepsilon_i\\ \beta-\varepsilon_i\end{bmatrix} q^{\mp\gamma_i} + q^{\pm\beta_i}\begin{bmatrix}\beta+\gamma-\varepsilon_i\\ \beta\end{bmatrix},$$

then by **2.4** (i), **3.1** (ii) & Lemma 2.1, we get

$$\begin{aligned} &\partial_i(x^{(\beta)})\,\sigma_i^{\mp1}(x^{(\gamma)}) + \big(\Theta(-\varepsilon_i)\sigma_i^{\pm1}\big)(x^{(\beta)})\,\partial_i(x^{(\gamma)}) \\ &\quad= q^{-\varepsilon_i*\beta\mp\gamma_i}\,x^{(\beta-\varepsilon_i)}\,x^{(\gamma)} + \theta(-\varepsilon_i,\beta)\,q^{\pm\beta_i-\varepsilon_i*\gamma}x^{(\beta)}\,x^{(\gamma-\varepsilon_i)} \\ &\quad= \Bigg(q^{-\varepsilon_i*\beta\mp\gamma_i+(\beta-\varepsilon_i)*\gamma}\begin{bmatrix}\beta+\gamma-\varepsilon_i\\ \beta-\varepsilon_i\end{bmatrix} \\ &\qquad\quad + q^{-\varepsilon_i*\beta+\beta*\varepsilon_i\pm\beta_i-\varepsilon_i*\gamma+\beta*(\gamma-\varepsilon_i)}\begin{bmatrix}\beta+\gamma-\varepsilon_i\\ \beta\end{bmatrix}\Bigg)\,x^{(\beta+\gamma-\varepsilon_i)} \\ &\quad= q^{-\varepsilon_i*(\beta+\gamma)+\beta*\gamma}\left(\begin{bmatrix}\beta+\gamma-\varepsilon_i\\ \beta-\varepsilon_i\end{bmatrix} q^{\mp\gamma_i} + q^{\pm\beta_i}\begin{bmatrix}\beta+\gamma-\varepsilon_i\\ \beta\end{bmatrix}\right) x^{(\beta+\gamma-\varepsilon_i)} \\ &\quad= q^{-\varepsilon_i*(\beta+\gamma)+\beta*\gamma}\begin{bmatrix}\beta+\gamma\\ \beta\end{bmatrix} x^{(\beta+\gamma-\varepsilon_i)} \\ &\quad= q^{\beta*\gamma}\begin{bmatrix}\beta+\gamma\\ \beta\end{bmatrix}\partial_i(x^{(\beta+\gamma)}) = \partial_i(x^{(\beta)}\,x^{(\gamma)}). \end{aligned}$$

Therefore, ∂_i is a $(\Theta(-\varepsilon_i)\sigma_i^{\pm1}, \sigma_i^{\mp1})$-derivation of $\mathcal{A}_q$ in the sense of **1.3** (iii).

(3) By (ii) and **2.1** (ii), for any $x^{(\gamma)}\in\mathcal{A}_q$, we get

$$\begin{aligned} \partial_i\,\partial_j(x^{(\gamma)}) &= q^{-(\varepsilon_i+\varepsilon_j)*\gamma+\varepsilon_i*\varepsilon_j}\,x^{(\gamma-\varepsilon_i-\varepsilon_j)} \\ &= \theta(\varepsilon_i,\varepsilon_j)\,q^{-(\varepsilon_i+\varepsilon_j)*\gamma+\varepsilon_j*\varepsilon_i}\,x^{(\gamma-\varepsilon_i-\varepsilon_j)} \\ &= \theta(\varepsilon_i,\varepsilon_j)\,\partial_j\,\partial_i(x^{(\gamma)}). \end{aligned}$$

(4) follows from the associativity and the θ-commutativity of $\mathcal{A}_q$.

(5) is clear.

(6) is obtained by combining (2) with (4), and observing the additivity of Θ in (1). $\square$

Remark. The commutative coefficients between ∂_i in Proposition 3.1 (3) coincide with those of $x^{(\varepsilon_i)} = x_i$, that is, $\partial_i\,\partial_j = \theta(\varepsilon_i,\varepsilon_j)\,\partial_j\,\partial_i$ and $x_i\,x_j = \theta(\varepsilon_i,\varepsilon_j)x_j\,x_i$. In the discussion of Wess & Zumino in [17], however, $x\,y = q\,y\,x$ but $\partial_x\,\partial_y = q^{-1}\partial_y\,\partial_x$ (cf. p. 309 [17], (4.5) & (4.7)), which distinguishes from our case. Especially, in process of our introducing the q-differential operators ∂_i over $\mathcal{A}_q$, of particular interest is that will lead to a new quantum group structure below (in the sense of Drinfeld).

3.2 Now let $\mathfrak{D}_q$ be the associative algebra over k generated by the symbols $\Theta(\pm\varepsilon_i)$, $\sigma_i^{\pm 1}$, ∂_i $(1 \le i \le n)$, associated to the bicharacter θ on $\mathbb{Z}^n$ given in **2.1**, satisfying the following relations:

$$\text{(i)} \qquad \sigma_i\sigma_i^{-1} = 1 = \sigma_i^{-1}\sigma_i, \qquad \sigma_i\sigma_j = \sigma_j\sigma_i,$$
$$\sigma_i\,\Theta(\varepsilon_j) = \Theta(\varepsilon_j)\,\sigma_i, \qquad \Theta(-\varepsilon_i+\varepsilon_{i+1}) = \sigma_i\sigma_{i+1},$$
$$\text{(ii)} \qquad \Theta(\varepsilon_i)^{-1} = \Theta(-\varepsilon_i), \qquad \Theta(0) = 1,$$
$$\Theta(\varepsilon_i)\,\Theta(\varepsilon_j) = \Theta(\varepsilon_i+\varepsilon_j) = \Theta(\varepsilon_j)\,\Theta(\varepsilon_i),$$
$$\text{(iii)} \qquad \Theta(\varepsilon_j)\,\partial_i\,\Theta(\varepsilon_j)^{-1} = \theta(\varepsilon_i,\varepsilon_j)\,\partial_i,$$
$$\text{(iv)} \qquad \sigma_j\,\partial_i\,\sigma_j^{-1} = q^{-\delta_{ij}}\,\partial_i,$$
$$\text{(v)} \qquad \partial_i\,\partial_j = \theta(\varepsilon_i,\varepsilon_j)\,\partial_j\,\partial_i.$$

Furthermore, $\mathfrak{D}_q$ can be equipped with a quantum group structure if we define the following mappings Δ, ϵ and S on the generators of $\mathfrak{D}_q$ as

$$\text{(vi)} \qquad \Delta : \mathfrak{D}_q \to \mathfrak{D}_q \otimes \mathfrak{D}_q$$
$$\Delta\,(\sigma_i^{\pm 1}) = \sigma_i^{\pm 1} \otimes \sigma_i^{\pm 1},$$
$$\Delta\,(\Theta(\pm\varepsilon_i)) = \Theta(\pm\varepsilon_i) \otimes \Theta(\pm\varepsilon_i),$$
$$\Delta\,(\partial_i) = \partial_i \otimes \sigma_i^{-1} + \Theta(-\varepsilon_i)\sigma_i \otimes \partial_i.$$
$$\text{(vii)} \qquad \epsilon : \mathfrak{D}_q \to k$$
$$\epsilon(\sigma_i^{\pm 1}) = 1 = \epsilon(\Theta(\pm\varepsilon_i)),$$
$$\epsilon(\partial_i) = 0.$$
$$\text{(viii)} \qquad S : \mathfrak{D}_q \to \mathfrak{D}_q$$
$$S(\sigma_i^{\pm 1}) = \sigma_i^{\mp 1},$$
$$S(\Theta(\pm\varepsilon_i)) = \Theta(\mp\varepsilon_i),$$
$$S(\partial_i) = -q\,\Theta(\varepsilon_i)\,\partial_i.$$

Again we extend the definitions of Δ, ϵ (resp. S) on $\mathfrak{D}_q$ (anti-)algebraically. Thus we obtain the following

Theorem. $(\mathfrak{D}_q, \Delta, \epsilon, S)$ *is a quantum group with the above relations* (i) — (viii).

Proof. First we need to show that Δ, ϵ and S preserve the algebraic relations (i)—(v) of $\mathfrak{D}_q$. This is clear for ϵ and S, and clear for Δ preserving relations (i)—(ii).

So it remains to check it for Δ with respect to relations (iii)—(v). Note that

$$\begin{aligned}(\Theta(-\varepsilon_i)\sigma_i)\,\partial_j &= \theta(\varepsilon_i,\varepsilon_j)\,\partial_j\,(\Theta(-\varepsilon_i)\sigma_i),\\ \partial_i\,(\Theta(-\varepsilon_j)\sigma_j) &= \theta(\varepsilon_i,\varepsilon_j)\,(\Theta(-\varepsilon_j)\sigma_j)\,\partial_i,\end{aligned}$$

for $i\neq j$. Hence, we have

$$\begin{aligned}\Delta(\Theta(\varepsilon_j))\Delta(\partial_i)\Delta(\Theta(\varepsilon_j)^{-1}) &= \Theta(\varepsilon_j)\,\partial_i\,\Theta(\varepsilon_j)^{-1}\otimes\sigma_i^{-1}\\ &\qquad+\Theta(-\varepsilon_i)\sigma_i\otimes\Theta(\varepsilon_j)\,\partial_i\,\Theta(\varepsilon_j)^{-1}\\ &= \theta(\varepsilon_i,\varepsilon_j)\Delta(\partial_i),\end{aligned}$$

$$\begin{aligned}\Delta(\sigma_j)\Delta(\partial_i)\Delta(\sigma_j^{-1}) &= \sigma_j\,\partial_i\,\sigma_j^{-1}\otimes\sigma_i^{-1}+\Theta(-\varepsilon_i)\sigma_i\otimes\sigma_j\,\partial_i\,\sigma_j^{-1}\\ &= q^{-\delta_{ij}}\Delta(\partial_i),\end{aligned}$$

$$\begin{aligned}\Delta(\partial_i)\Delta(\partial_j) &= (\partial_i\otimes\sigma_i^{-1}+\Theta(-\varepsilon_i)\sigma_i\otimes\partial_i)(\partial_j\otimes\sigma_j^{-1}+\Theta(-\varepsilon_j)\sigma_j\otimes\partial_j)\\ &= \partial_i\partial_j\otimes\sigma_i^{-1}\sigma_j^{-1}+\Theta(-\varepsilon_i-\varepsilon_j)\sigma_i\sigma_j\otimes\partial_i\partial_j\\ &\qquad+\,(\Theta(-\varepsilon_i)\sigma_i)\,\partial_j\otimes\partial_i\sigma_j^{-1}+\partial_i\,(\Theta(-\varepsilon_j)\sigma_j)\otimes\sigma_i^{-1}\partial_j\\ &= \theta(\varepsilon_i,\varepsilon_j)\big(\partial_j\partial_i\otimes\sigma_j^{-1}\sigma_i^{-1}+\Theta(-\varepsilon_i-\varepsilon_j)\sigma_j\sigma_i\otimes\partial_j\partial_i\\ &\qquad+\,(\Theta(-\varepsilon_j)\sigma_j)\,\partial_i\otimes\partial_j\sigma_i^{-1}+\partial_j\,(\Theta(-\varepsilon_i)\sigma_i)\otimes\sigma_j^{-1}\partial_i\big)\\ &= \theta(\varepsilon_i,\varepsilon_j)\Delta(\partial_j)\Delta(\partial_i),\qquad(i\neq j).\end{aligned}$$

In view of the fact just proved, together with (vi) & (vii), we see that $(1\otimes\Delta)\Delta=(\Delta\otimes 1)\Delta$ and $(1\otimes\epsilon)\Delta=1=(\epsilon\otimes 1)\Delta$ hold. Again by (vi) & (viii), we have $m\circ(1\otimes S)\circ\Delta(\partial_i)=\partial_i\sigma_i+\Theta(-\varepsilon_i)\,\sigma_i\,(-q\Theta(\varepsilon_i)\partial_i)=\partial_i\sigma_i-q\sigma_i\partial_i=0$ and $m\circ(S\otimes 1)\circ\Delta(\partial_i)=-q\Theta(\varepsilon_i)\,\partial_i\,\sigma_i^{-1}+\sigma_i^{-1}\,\Theta(\varepsilon_i)\,\partial_i=\Theta(\varepsilon_i)(-q\partial_i\sigma_i^{-1}+\sigma_i^{-1}\partial_i)=0$, thus we get $m\circ(1\otimes S)\circ\Delta(\partial_i)=\epsilon(\partial_i)=m\circ(S\otimes 1)\circ\Delta(\partial_i)$. On the other hand, owing to (vi) and $\Theta(\pm\varepsilon_i)\,\Theta(\mp\varepsilon_i)=1=\sigma_i^{\pm1}\sigma_i^{\mp1}$, there holds $m\circ(1\otimes S)\circ\Delta=\eta\circ\epsilon=m\circ(S\otimes 1)\circ\Delta$, where $(\mathfrak{D}_q,m,\eta)$ is the algebra structure of $\mathfrak{D}_q$.

Thereby, $(\mathfrak{D}_q,m,\eta,\Delta,\epsilon,S)$ is a non-commutative and non-cocommutative Hopf algebra, namely, a quantum group. $\Box$

Remark. Actually, we can equip $\mathfrak{D}_q$ with another quantum group structure ($\mathfrak{D}_q$, $\Delta^{(-)}$, ϵ, $S^{(-)}$), where only one difference is the actions of $\Delta^{(-)}$ and $S^{(-)}$ on ∂_i $(1\le i\le n)$ given respectively by $\Delta^{(-)}(\partial_i)=\partial_i\otimes\sigma_i+\Theta(-\varepsilon_i)\sigma_i^{-1}\otimes\partial_i$, and $S^{(-)}(\partial_i)=-q^{-1}\,\Theta(\varepsilon_i)\,\partial_i$.

3.3 Based on the structure of quantum group $\mathfrak{D}_q$ in Theorem 3.2 and observing that the commutative rule for x_i in $\mathcal{A}_q$ is the same as that of the special q-derivatives ∂_i (see Remark 3.1), we can augment $\mathcal{A}_q$ through adding a certain multiplication abelian group Θ (as group-like elements), and construct another quantum group $\mathfrak{A}_q$ below such that it contains the quantum divided power algebra $\mathcal{A}_q$ as its subalgebra. Here $\Theta=\{\Theta(\alpha)\mid\alpha\in\mathbb{Z}^n\}$ acts conjugately on $\mathcal{A}_q$ as its an automorphism group. When $\mathcal{A}_q$ (as an object in the braided category $\mathcal{GB}$) is of a braided Hopf algebra

structure, we has a reasonable interpretation for such a construction of the quantum group $\mathfrak{A}_q$ (as an object in the usual category $\mathcal{HA}$ of Hopf algebras): the price introducing the group Θ into $\mathcal{A}_q$ lies in transmuting the "braided" twisting of the algebra structure on $\mathcal{A}_q\otimes\mathcal{A}_q$ (in $\mathcal{GB}$) into the "trivial" twisting of the algebra structure on $\mathfrak{A}_q\otimes\mathfrak{A}_q$ (in $\mathcal{HA}$). As is clear, according to their respective comultiplications (as algebra homomorphisms!).

Let $\mathfrak{A}_q$ be the associative algebra over k generated by the quantum divided power (restricted) algebra $\mathcal{A}_q$, together with the symbols $\Theta(\pm\varepsilon_i)$ $(1\le i\le n)$, associated to the bicharacter θ on $\mathbb{Z}^n$ given in **2.1**, subject to the relations

$$\text{(i)}\qquad \Theta(\varepsilon_i)^{-1}=\Theta(-\varepsilon_i),\qquad \Theta(0)=1,$$
$$\Theta(\varepsilon_i)\,\Theta(\varepsilon_j)=\Theta(\varepsilon_i+\varepsilon_j)=\Theta(\varepsilon_j)\,\Theta(\varepsilon_i),$$
$$\text{(ii)}\qquad \Theta(\varepsilon_j)\,x_i\,\Theta(\varepsilon_j)^{-1}=\theta(\varepsilon_j,\varepsilon_i)\,x_i,$$
$$\big(\,\Theta(\varepsilon_j)\,x_i^{(l)}\,\Theta(\varepsilon_j)^{-1}=x_i^{(l)}\quad \textit{in } \mathfrak{A}_q(n)\ \textit{if in addition } q^l=1\,\big)$$
$$\text{(iii)}\qquad x_i\,x_j=\theta(\varepsilon_i,\varepsilon_j)\,x_j\,x_i.$$
$$\big(\,x_i\,x_j^{(l)}=x_j^{(l)}\,x_i\quad \textit{in } \mathfrak{A}_q(n)\ \textit{if in addition } q^l=1\,\big)$$

Note that $\mathfrak{A}_q$ here also indicates $\mathfrak{A}_q(n)$ for any $\mathbf{char}(q)$ or $\mathfrak{A}_q(n,\mathbf{1})$ only for $q^l=1$. Similar to Theorem 3.2, we have

Theorem. $(\mathfrak{A}_q,\Delta,\epsilon,S)$ *forms a quantum group, which is the required quantization object of the polynomial algebra in n variables in the context of quantum groups when* $\mathbf{char}(q)=0$ (*also can be viewed as the quantized universal enveloping algebra of the abelian Lie algebra of dimension n*)*, under the comultiplication Δ, the counit ϵ and the antipode S given by*

$$\text{(iv)}\qquad \Delta:\mathfrak{A}_q\to\mathfrak{A}_q\otimes\mathfrak{A}_q$$
$$\Delta\,(\Theta(\pm\varepsilon_i))=\Theta(\pm\varepsilon_i)\otimes\Theta(\pm\varepsilon_i),$$
$$\Delta\,(x_i)=x_i\otimes 1+\Theta(\varepsilon_i)\otimes x_i.$$
$$\big(\,\Delta\,(x_i^{(l)})=x_i^{(l)}\otimes 1+1\otimes x_i^{(l)}\quad \textit{in } \mathfrak{A}_q(n)\ \textit{only for } q^l=1\,\big)$$
$$\text{(v)}\qquad \epsilon:\mathfrak{A}_q\to k$$
$$\epsilon(\Theta(\pm\varepsilon_i))=1,$$
$$\epsilon(x_i)=0.$$
$$\big(\,\epsilon(x_i^{(l)})=0\quad \textit{in } \mathfrak{A}_q(n)\ \textit{only for } q^l=1\,\big)$$
$$\text{(vi)}\qquad S:\mathfrak{A}_q\to\mathfrak{A}_q$$
$$S(\Theta(\pm\varepsilon_i))=\Theta(\mp\varepsilon_i),$$
$$S(x_i)=-\Theta(-\varepsilon_i)\,x_i.$$
$$\big(\,S(x_i^{(l)})=-x_i^{(l)}\quad \textit{in } \mathfrak{A}_q(n)\ \textit{only for } q^l=1\,\big)$$

Proof. It follows from a similar argument as Theorem 3.2. $\square$

Remark. Given a Hopf algebra (H,Δ,ϵ,S), for $a\in H$, consider its *left adjoint action* $\mathrm{ad}\,a$ on H: $\mathrm{ad}\,a(b)=a_1bS(a_2)$ where $\Delta(a)=a_1\otimes a_2$ (in Sweedler convention). Clearly, the map $a\mapsto \mathrm{ad}\,a$ of H into $\mathrm{End}\,H$ is an algebra homomorphism. Now for $H=\mathfrak{A}_q$, consider the action of $\mathrm{ad}\,x_i$ on its subalgebra $\mathcal{A}_q$, that is, $\forall\, x^{(\gamma)}\in\mathcal{A}_q$, we have $\mathrm{ad}\,x_i(x^{(\gamma)})=x_i\,x^{(\gamma)}-\Theta(\varepsilon_i)\,x^{(\gamma)}\,\Theta(-\varepsilon_i)\,x_i=x_i\,x^{(\gamma)}-\theta(\varepsilon_i,\gamma)\,x^{(\gamma)}\,x_i=0$. On the other hand, if in addition $q^l=1$, we get $\mathrm{ad}\,x_i^{(l)}(x^{(\gamma)})=x_i^{(l)}\,x^{(\gamma)}-x^{(\gamma)}\,x_i^{(l)}=0$ in $\mathcal{A}_q(n)$. So $\mathrm{ad}\,x^{(\alpha)}|_{\mathcal{A}_q}=0$ for any $x^{(\alpha)}\in\mathcal{A}_q$. This fact is compatible with the classical case, since the group Θ degenerates into the unit group and $\mathfrak{A}_q(n)$ into a polynomial algebra in n variables with a known Hopf algebra structure when q takes 1. Consequently, in the case when $\mathbf{char}(q)=0$, the quantum group $\mathfrak{A}_q(n)$ achieved above is just the corresponding *object* of the polynomial algebra in n variables in the context of quantum groups, which also can be considered as the quantized universal enveloping algebra of the abelian Lie algebra of dimension n.

3.4 By analogy of the argument in **3.2**, we denote $\mathfrak{U}_q$ by the associative algebra over k generated by the symbols $\Theta(\pm\varepsilon_i)$, $\sigma_i^{\pm1}$, x_i $(1\le i\le n)$, associated to the bicharacter θ on $\mathbb{Z}^n$ given in **2.1**, satisfying the following relations:

(i)
$$\sigma_i\sigma_i^{-1}=1=\sigma_i^{-1}\sigma_i,\qquad \sigma_i\sigma_j=\sigma_j\sigma_i,$$
$$\sigma_i\,\Theta(\varepsilon_j)=\Theta(\varepsilon_j)\,\sigma_i,\qquad \Theta(-\varepsilon_i+\varepsilon_{i+1})=\sigma_i\sigma_{i+1},$$

(ii)
$$\Theta(\varepsilon_i)^{-1}=\Theta(-\varepsilon_i),\qquad \Theta(0)=1,$$
$$\Theta(\varepsilon_i)\,\Theta(\varepsilon_j)=\Theta(\varepsilon_i+\varepsilon_j)=\Theta(\varepsilon_j)\,\Theta(\varepsilon_i),$$
$$\big(\,\Theta(\varepsilon_i)^l=\Theta(l\,\varepsilon_i)=1,\qquad \sigma_i^l=1\quad \textit{only when}\quad q^l=1\,\big)$$

(iii)
$$\Theta(\varepsilon_i)\,x_j\,\Theta(\varepsilon_i)^{-1}=\theta(\varepsilon_i,\varepsilon_j)\,x_j,$$

(iv)
$$\sigma_i\,x_j\,\sigma_i^{-1}=q^{\delta_{ij}}\,x_j,$$

(v)
$$x_i\,x_j=\theta(\varepsilon_i,\varepsilon_j)\,x_j\,x_i.$$

Moreover, the comultiplication Δ, the counity ϵ and the antipode S over $\mathfrak{U}_q$ are defined respectively by

(vi)
$$\Delta:\mathfrak{U}_q\to\mathfrak{U}_q\otimes\mathfrak{U}_q$$
$$\Delta\,(\sigma_i^{\pm1})=\sigma_i^{\pm1}\otimes\sigma_i^{\pm1},$$
$$\Delta\,(\Theta(\pm\varepsilon_i))=\Theta(\pm\varepsilon_i)\otimes\Theta(\pm\varepsilon_i),$$
$$\Delta\,(x_i)=x_i\otimes\sigma_i+\Theta(\varepsilon_i)\sigma_i^{-1}\otimes x_i.$$

(vii)
$$\epsilon:\mathfrak{U}_q\to k$$
$$\epsilon(\sigma_i^{\pm1})=1=\epsilon(\Theta(\pm\varepsilon_i)),$$
$$\epsilon(x_i)=0.$$

(viii)
$$S:\mathfrak{U}_q\to\mathfrak{U}_q$$
$$S(\sigma_i^{\pm1})=\sigma_i^{\mp1},$$
$$S(\Theta(\pm\varepsilon_i))=\Theta(\mp\varepsilon_i),$$
$$S(x_i)=-q\,\Theta(-\varepsilon_i)\,x_i.$$

Similar to Theorem 3.2, we obtain

Theorem. $(\mathfrak{U}_q, \Delta, \epsilon, S)$ *is the quantum group corresponding to the quantum n-space $k[A_q^{n|0}]$ in particular, in the case when* $\mathbf{char}(q) = l$ *is odd, whose Hopf algebra structure restricted on its central subalgebra $k[x_1^l, \cdots, x_n^l]$ is just the usual Hopf algebra structure of the polynomial algebra in n variables.*

Proof. The first claim follows from a similar argument as Theorem 3.2.

On the other hand, using formula (vi) and induction on m, we get

$$\Delta(x_i^m) = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} x_i^{m-k}\, \Theta(\varepsilon_i)^k\, \sigma_i^{-k} \otimes x_i^k\, \sigma_i^{m-k}.$$

In particular, since $\mathbf{char}(q) = l$ being odd means $q^l = 1$, we have $\left[\begin{smallmatrix} l \\ k \end{smallmatrix}\right] = 0$ for $1 \le k < l$ and $\Theta(\varepsilon_i)^l\, \sigma_i^{-l} = 1$ so that $\Delta(x_i^l) = x_i^l \otimes 1 + 1 \otimes x_i^l$ and $S(x_i^l) = -x_i^l$. These are compatible with the fact that $k[x_1^l, \cdots, x_n^l]$ is as a (polynomial) central subalgebra of $k[A_q^{n|0}]$ when $q^l = 1$. $\square$

Remark. Actually, since the fact that $x_i\,(1 \le i \le n)$ are the generators of $k[A_q^{n|0}]$ is independently of the *characteristic* $\mathbf{char}(q)$ of q. Thereby, when q is a primitive l-th root of unity, the quantum group object corresponding to the quantum n-space $k[A_q^{n|0}]$ can be directly defined by the same relations (i) — (viii) as in **3.4**.

3.5 Let $\mathfrak{D}_q^{(\pm)} := (\mathfrak{D}_q, \Delta^{(\pm)}, \epsilon, S^{(\pm)})$ denote the two Hopf algebras appearing in **3.2**. Then from Proposition 3.1 (2) & (5), we see that $\mathcal{A}_q$ is a (left) $\mathfrak{D}_q^{(\pm)}$-module algebra. So we are able to make their smash product algebras $\mathcal{A}_q \# \mathfrak{D}_q^{(\pm)}$ in a familiar fashion as in [16], which are the same as $\mathcal{A}_q \otimes \mathfrak{D}_q$ as vector spaces but multiplications respectively given by

$$\begin{aligned}
(x^{(\alpha)} \# \partial_i) \circ (x^{(\beta)} \# d) &= x^{(\alpha)}\, \partial_i(x^{(\beta)}) \# \sigma_i^{\mp 1}\, d + x^{(\alpha)} (\Theta(-\varepsilon_i)\sigma_i^{\pm 1})(x^{(\beta)}) \# \partial_i\, d \\
&= q^{\alpha * \beta - \alpha * \varepsilon_i - \varepsilon_i * \beta} \begin{bmatrix} \alpha + \beta - \varepsilon_i \\ \alpha \end{bmatrix} x^{(\alpha+\beta-\varepsilon_i)} \# \sigma_i^{\mp 1}\, d \\
&\quad + \theta(\beta, \varepsilon_i)\, q^{\alpha*\beta \pm \beta_i} \begin{bmatrix} \alpha + \beta \\ \alpha \end{bmatrix} x^{(\alpha+\beta)} \# \partial_i\, d, \\
(x^{(\alpha)} \# g) \circ (x^{(\beta)} \# d) &= x^{(\alpha)}\, g(x^{(\beta)}) \# g\, d,
\end{aligned}$$

where $\Delta^{(\pm)}(\partial_i) = \partial_i \otimes \sigma_i^{\mp 1} + \Theta(-\varepsilon_i)\, \sigma_i^{\pm 1} \otimes \partial_i$ and $\Delta^{(\pm)}(g) = g \otimes g$, for $x^{(\alpha)}, x^{(\beta)} \in \mathcal{A}_q$ and $\partial_i, g, d \in \mathfrak{D}_q$. More precisely, we have

$$\text{(i)} \qquad \begin{aligned}
(x^{(\alpha)} \# 1) \circ (1 \# d) &= x^{(\alpha)} \# d, \\
(1 \# \Theta(\varepsilon_i)) \circ (x_j \# 1) &= \Theta(\varepsilon_i)(x_j) \# \Theta(\varepsilon_i) \\
&= \theta(\varepsilon_i, \varepsilon_j)\, x_j \# \Theta(\varepsilon_i), \\
(1 \# \sigma_i) \circ (x_j \# 1) &= q^{\delta_{ij}}\, x_j \# \sigma_i, \\
(1 \# \partial_i) \circ (x_j \# 1) &= \delta_{ij} \# \sigma_i^{\mp 1} + \theta(\varepsilon_j, \varepsilon_i)\, q^{\pm \delta_{ij}}\, x_j \# \partial_i.
\end{aligned}$$

Actually, if we briefly identify elements $x^{(\alpha)} \# d$ in $\mathcal{A}_q \# \mathfrak{D}_q^{(\pm)}$ with $x^{(\alpha)}\, d$, then the smash product algebras $\mathcal{A}_q \# \mathfrak{D}_q^{(\pm)}$ containing $\mathcal{A}_q$ and $\mathfrak{D}_q^{(\pm)}$ as subalgebras are just

the *quantum differential operators algebras* over $\mathcal{A}_q$, which will degenerate into the usual differential operators algebra when q takes 1. Particularly, combining formulae (i) with the identification above, we get the relations below

$$\text{(ii)} \qquad \begin{aligned} x^{(\alpha)} \circ d &= x^{(\alpha)}\, d, \\ \Theta(\varepsilon_i) \circ x_j \circ \Theta(\varepsilon_i)^{-1} &= \theta(\varepsilon_i, \varepsilon_j)\, x_j, \\ \sigma_i \circ x_j \circ \sigma_i^{-1} &= q^{\delta_{ij}}\, x_j, \\ \partial_i \circ x_j &= \delta_{ij}\sigma_i^{\mp 1} + \theta(\varepsilon_j, \varepsilon_i)\, q^{\pm\delta_{ij}} x_j \circ \partial_i. \end{aligned}$$

Applying formulae (ii), together with $\mathcal{A}_q$ and $\mathfrak{D}_q^{(\pm)}$, we are able to construct the following *quantum Weyl algebra*, which is different from those appeared in the literature for instance, Proposition 5.2.2 in [3], Section 2.1 in [5], [10], etc.).

Definition. *Let $\mathcal{W}_q(2n)$ be the associative algebra over k generated by the symbols $\Theta(\pm\varepsilon_i)$, $\sigma_i^{\pm 1}$, x_i and ∂_i $(1 \le i \le n)$, associated to the bicharacter θ on $\mathbb{Z}^n$ defined in* **2.1***, obeying the following relations:*

$$\begin{aligned}
&\text{(iii)} && \Theta(\pm\varepsilon_i) \circ \Theta(\mp\varepsilon_i) = 1 = \sigma_i^{\pm 1} \circ \sigma_i^{\mp 1}, \\
& && \Theta(-\varepsilon_i + \varepsilon_{i+1}) = \sigma_i \circ \sigma_{i+1}, \\
& && \Theta(\varepsilon_i) \circ \Theta(\varepsilon_j) = \Theta(\varepsilon_i + \varepsilon_j) = \Theta(\varepsilon_j) \circ \Theta(\varepsilon_i), \\
& && \sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i, \\
& && \sigma_i \circ \Theta(\varepsilon_j) = \Theta(\varepsilon_j) \circ \sigma_i, \\
&\text{(iv)} && \Theta(\varepsilon_i) \circ x_j \circ \Theta(-\varepsilon_i) = \theta(\varepsilon_i, \varepsilon_j)\, x_j, \\
& && \Theta(\varepsilon_i) \circ \partial_j \circ \Theta(-\varepsilon_i) = \theta(\varepsilon_j, \varepsilon_i)\, \partial_j, \\
&\text{(v)} && \sigma_i \circ x_j \circ \sigma_i^{-1} = q^{\delta_{ij}}\, x_j, \\
& && \sigma_i \circ \partial_j \circ \sigma_i^{-1} = q^{-\delta_{ij}}\, \partial_j, \\
&\text{(vi)} && x_i \circ x_j = \theta(\varepsilon_i, \varepsilon_j)\, x_j \circ X_i, \\
& && \partial_i \circ \partial_j = \theta(\varepsilon_i, \varepsilon_j)\, \partial_j \circ \partial_i, \\
&\text{(vii)} && \partial_i \circ x_j = \theta(\varepsilon_j, \varepsilon_i)\, x_j \circ \partial_i, \qquad (i \ne j), \\
&\text{(viii)} && \partial_i \circ X_i - q^{\pm 1} x_i \circ \partial_i = \sigma_i^{\mp 1}.
\end{aligned}$$

where the relations (viii) *are equivalent to the following relations:*

$$\text{(ix)} \qquad \partial_i \circ x_i = \frac{q\,\sigma_i - (q\,\sigma_i)^{-1}}{q - q^{-1}}, \qquad x_i \circ \partial_i = \frac{\sigma_i - \sigma_i^{-1}}{q - q^{-1}}.$$

Remark. The relations above imply $\mathcal{W}_q(2n)$ also contains $\mathfrak{U}_q$ given in **3.4** as its subalgebra. However, $\mathcal{W}_q(2n)$ itself doesn't to be a Hopf algebra due to the last relations (viii), which is coincident with the classical situation.

4. An Application: Realization

4.1 As an application of discussions in section **3**, we are now in the position to realize the quantized universal enveloping algebra $U_q(\mathfrak{g})$ (where $\mathfrak{g}=\mathfrak{gl}_n$ or $\mathfrak{sl}_n$) as certain q-differential operators in $\mathcal{W}_q(2n)$ defined over the quantum divided power (restricted) algebra $\mathcal{A}_q$, such that the quantum divided power (restricted) algebra $\mathcal{A}_q$ is made into a $U_q(\mathfrak{g})$-module algebra in the sense of **1.3**.

Theorem. *For any monomial $x^{(\beta)}\in\mathcal{A}_q$ and $1\le i<n$, set*

$$\text{(i)}\qquad e_i(x^{(\beta)})=\big(\,x_i\,\partial_{i+1}\,\sigma_i\,\big)\,(x^{(\beta)}),$$

$$\text{(ii)}\qquad f_i(x^{(\beta)})=\big(\,\sigma_i^{-1}\,x_{i+1}\,\partial_i\,\big)\,(x^{(\beta)}),$$

$$\text{(iii)}\qquad \mathcal{K}_i(x^{(\beta)})=(\sigma_i\,\sigma_{i+1}^{-1})\,(x^{(\beta)}),$$

$$\text{(iv)}\qquad \mathcal{K}_i^{-1}(x^{(\beta)})=(\sigma_i^{-1}\,\sigma_{i+1})\,(x^{(\beta)}).$$

Formulas (i)–(iv) *define the structure of a $U_q(\mathfrak{sl}_n)$-module algebra on $\mathcal{A}_q$.* $\square$

Proof. The proof will be given in two steps.

(I) We first show that the formulas (i)–(iv) equip $\mathcal{A}_q$ with a $U_q(\mathfrak{sl}_n)$-module structure. To do this, we need to check the algebra relations **1.4** (i)–(v) of $U_q(\mathfrak{sl}_n)$.

Using Lemma 2.1 (2), **2.4** (i) & **3.1** (ii), we get from relations (i) & (ii) that

$$\text{(v)}\qquad \begin{aligned} e_i(x^{(\beta)})&=q^{\beta_i-\varepsilon_{i+1}*\beta+\varepsilon_i*(\beta-\varepsilon_{i+1})}\begin{bmatrix}\beta+\varepsilon_i-\varepsilon_{i+1}\\ \varepsilon_i\end{bmatrix}x^{(\beta+\varepsilon_i-\varepsilon_{i+1})}\\ &=[\,\beta_i+1\,]\,x^{(\beta+\varepsilon_i-\varepsilon_{i+1})},\\ f_i(x^{(\beta)})&=q^{-\varepsilon_i*\beta+\varepsilon_{i+1}*(\beta-\varepsilon_i)-(\beta_i-1)}\begin{bmatrix}\beta-\varepsilon_i+\varepsilon_{i+1}\\ \varepsilon_{i+1}\end{bmatrix}x^{(\beta-\varepsilon_i+\varepsilon_{i+1})}\\ &=[\,\beta_{i+1}+1\,]\,x^{(\beta-\varepsilon_i+\varepsilon_{i+1})}.\end{aligned}$$

Clearly, relation **1.4** (i) holds. For relation **1.4** (ii), we have

$$\begin{aligned}\mathcal{K}_i\,e_j\,\mathcal{K}_i^{-1}(x^{(\beta)})&=q^{-\beta_i+\beta_{i+1}}\,[\beta_j+1]\,\mathcal{K}_i\,(x^{(\beta+\varepsilon_j-\varepsilon_{j+1})})\\ &=q^{\delta_{ij}-\delta_{i,j+1}-\delta_{i+1,j}+\delta_{i+1,j+1}}\,e_j(x^{(\beta)}).\end{aligned}$$

We can show $\mathcal{K}_i\,f_j\,\mathcal{K}_i^{-1}=q^{-a_{ij}}f_j$ in a similar fashion. For relation **1.4** (iii) we have

$$\begin{aligned}[e_i,f_j]\,(x^{(\beta)})&=[\beta_{j+1}+1]\;e_i(x^{(\beta-\varepsilon_j+\varepsilon_{j+1})})-[\beta_i+1]\;f_j(x^{(\beta+\varepsilon_i-\varepsilon_{i+1})})\\ &=\big(\,[\beta_{j+1}+1]\;[\beta_i+1-\delta_{j,i}+\delta_{j+1,i}]\\ &\qquad-[\beta_i+1]\;[\beta_{j+1}+1+\delta_{i,j+1}-\delta_{i+1,j+1}]\,\big)\;x^{(\beta+\varepsilon_i+\varepsilon_{j+1}-\varepsilon_{i+1}-\varepsilon_j)}\\ &=\delta_{ij}\big(\,[\beta_{i+1}+1]\;[\beta_i]-[\beta_i+1]\;[\beta_{i+1}]\,\big)\;x^{(\beta)}\\ &=\delta_{ij}\,\frac{q^{\beta_i-\beta_{i+1}}-q^{\beta_{i+1}-\beta_i}}{q-q^{-1}}\;x^{(\beta)}\\ &=\delta_{ij}\,\frac{\mathcal{K}_i-\mathcal{K}_i^{-1}}{q-q^{-1}}\;(x^{(\beta)}).\end{aligned}$$

As for relation **1.4** (iv), if $|i-j|>1$, then from formula (v) of e_i it is easy to see that $e_ie_j=e_je_i$. If $|i-j|=1$, without loss of generality, assume that $j=i+1$, then we shall check that

$$(*)\qquad \big(\, e_i^2\, e_{i+1}-(q+q^{-1})\, e_i\, e_{i+1}\, e_i+e_{i+1}\, e_i^2\,\big)\,(x^{(\beta)})=0.$$

Observing that

$$\begin{aligned} e_i^2\; e_{i+1}(x^{(\beta)}) &= [\beta_{i+1}+1]\;[\beta_i+1]\;[\beta_i+2]\,x^{(\beta+2\varepsilon_i-\varepsilon_{i+1}-\varepsilon_{i+2})},\\ e_i\; e_{i+1}\; e_i(x^{(\beta)}) &= [\beta_i+1]\;[\beta_{i+1}]\;[\beta_i+2]\,x^{(\beta+2\varepsilon_i-\varepsilon_{i+1}-\varepsilon_{i+2})},\\ e_{i+1}\; e_i^2(x^{(\beta)}) &= [\beta_i+1]\;[\beta_i+2]\;[\beta_{i+1}-1]\,x^{(\beta+2\varepsilon_i-\varepsilon_{i+1}-\varepsilon_{i+2})}, \end{aligned}$$

and $[m+1]-(q+q^{-1})\;[m]+[m-1]=0$, we see that the equality (*) holds. Similarly, we can prove

$$\big(\, e_{i+1}^2\; e_i-(q+q^{-1})\; e_{i+1}\; e_i\; e_{i+1}+e_i\; e_{i+1}^2\,\big)\;(x^{(\beta)})=0.$$

As for the last relation **1.4** (v), it is clear from formula (v) that $f_if_j=f_jf_i$ if $|i-j|>1$. When $|i-j|=1$, by (v), we have

$$\begin{aligned} &\big(\, f_{i+1}^2\; f_i-(q+q^{-1})\; f_{i+1}\; f_i\; f_{i+1}+f_i\; f_{i+1}^2\,\big)\;(x^{(\beta)})\\ &\qquad=\big(\,[\beta_{i+1}+1]\;[\beta_{i+2}+1]\;[\beta_{i+2}+2]\\ &\qquad\qquad-\;(q+q^{-1})\,[\beta_{i+2}+1]\;[\beta_{i+1}]\;[\beta_{i+2}+2]\\ &\qquad\qquad+\;[\beta_{i+2}+1]\;[\beta_{i+2}+2]\;[\beta_{i+1}-1]\,\big)\,x^{(\beta-\varepsilon_i-\varepsilon_{i+1}+2\varepsilon_{i+2})}=0. \end{aligned}$$

Similarly, we can check that $\big(\, f_i^2\; f_{i+1}-(q+q^{-1})\; f_i\; f_{i+1}\; f_i+f_{i+1}\; f_i^2\,\big)\;(x^{(\beta)})=0$.

(II) We next prove that the quantum divided power (restricted) algebra $\mathcal{A}_q$ is a $U_q(\mathfrak{sl}_n)$-algebra. By **1.3** (i) & (ii), **1.4** (vi)–(viii) and noting that the definition of $\mathcal{K}_i^{\pm}$ in (iii)–(iv), we need to check that for any $u\in U_q(\mathfrak{sl}_n)$, there hold

$$(1)\qquad u\,1=\epsilon(u)\;1,$$

$$(2)\qquad \mathcal{K}_i(x^{(\beta)}\;x^{(\gamma)})=\mathcal{K}_i(x^{(\beta)})\;\mathcal{K}_i(x^{(\gamma)}),$$

$$(3)\qquad e_i(x^{(\beta)}\;x^{(\gamma)})=x^{(\beta)}\;e_i(x^{(\gamma)})+e_i(x^{(\beta)})\;\mathcal{K}_i(x^{(\gamma)}),$$

$$(4)\qquad f_i(x^{(\beta)}\;x^{(\gamma)})=\mathcal{K}_i^{-1}(x^{(\beta)})\;f_i(x^{(\gamma)})+f_i(x^{(\beta)})\;x^{(\gamma)},$$

for any monomials $x^{(\beta)},x^{(\gamma)}\in\mathcal{A}_q$. Relation (1) follows easily from relations (i)–(iii) and **3.1** (i). Relation (2) is due to the fact that $\mathcal{K}_i$ acts as an algebra automorphism of $\mathcal{A}_q$.

By **1.3** (i)–(ii) & (iii), we shall prove that the endomorphism e_i $(1\le i<n)$ is a $(\mathrm{id},\sigma_i\sigma_{i+1}^{-1})$-derivation and f_i $(1\le i<n)$ is a $(\sigma_i^{-1}\sigma_{i+1},\mathrm{id})$-derivation, which implies relations (3) and (4).

Noting that $\Theta(\varepsilon_{i+1}-\varepsilon_i)=\sigma_i\,\sigma_{i+1}$ and using Proposition 3.1 (6), we see that $x^{(\varepsilon_i)}\partial_{i+1}$ is a $(\Theta(\varepsilon_i-\varepsilon_{i+1})\,\sigma_{i+1},\sigma_{i+1}^{-1})$-derivative of $\mathcal{A}_q$, i.e. $(\sigma_i^{-1},\sigma_{i+1}^{-1})$-derivative, hence, $x^{(\varepsilon_i)}\,\partial_{i+1}\,\sigma_i$ is a $(\mathrm{id},\sigma_i\,\sigma_{i+1}^{-1})$-derivative (since σ_i is an automorphism of $\mathcal{A}_q$). On the other hand, $x^{(\varepsilon_{i+1})}\partial_i$ is a $(\Theta(\varepsilon_{i+1}-\varepsilon_i)\sigma_i^{-1},\sigma_i)$-derivative of $\mathcal{A}_q$, i.e. (σ_{i+1},σ_i)-derivative, so $\sigma_i^{-1}\,x^{(\varepsilon_{i+1})}\,\partial_i$ is a $(\sigma_i^{-1}\,\sigma_{i+1},\mathrm{id})$-derivative (since σ_i^{-1} is an automorphism of $\mathcal{A}_q$).

Therefore, we complete the proof of Theorem 4.1. $\square$

Corollary. *For any monomial $x^{(\beta)}\in\mathcal{A}_q$, set*

$$
\begin{aligned}
e_i(x^{(\beta)}) &= \big(\,x_i\partial_{i+1}\sigma_i\,\big)\,(x^{(\beta)}) = [\beta_i+1]\,x^{(\beta+\varepsilon_i-\varepsilon_{i+1})} \qquad (1\le i<n),\\
f_i(x^{(\beta)}) &= \big(\,\sigma_i^{-1}x_{i+1}\partial_i\,\big)\,(x^{(\beta)}) = [\beta_{i+1}+1]\,x^{(\beta-\varepsilon_i+\varepsilon_{i+1})} \qquad (1\le i<n),\\
k_i(x^{(\beta)}) &= \sigma_i(x^{(\beta)}) \qquad (1\le i\le n),\\
k_i^{-1}(x^{(\beta)}) &= \sigma_i^{-1}(x^{(\beta)}) \qquad (1\le i\le n).
\end{aligned}
$$

Formulas (i)–(iv) *define the structure of a $U_q(\mathfrak{gl}_n)$-module algebra on $\mathcal{A}_q$.* □

Remark. According to the interpretation given in section **2.4**, the realization above is also valid for the quantum n-space $k[A_q^{n|0}]$, which can be viewed as an improvement of Hayashi's one (cf. [5] & [6]). It should be noticed that Hayashi's realization of $U_q(\mathfrak{sl}_n)$ was carried out over the usual polynomial algebra in n variables and we can verify that his realization cannot make the polynomial algebra a $U_q(\mathfrak{sl}_n)$-module algebra in the sense of **1.3** (cf. [1] & [16]).

4.2 As a direct consequence of Theorem 4.1 or Corollary 4.1, we consider the submodule structures of $\mathcal{A}_q$, where $\mathcal{A}_q=\mathcal{A}_q(n)$ if $\mathbf{char}(q)=0$, $\mathcal{A}_q=\mathcal{A}_q(n,\mathbf{1})$ if $\mathbf{char}(q)=l\,(\ge 3)$. Denote $|\,\alpha\,|:=\sum_{i=1}^n\alpha_i$ by the degree of $x^{(\alpha)}\in\mathcal{A}_q$, set $N:=|\,\tau\,|=n\,(l-1)$. Let $\mathcal{A}_q^{(s)}:=\langle\,x^{(\alpha)}\mid|\,\alpha\,|=s\,\rangle$. Then $\mathcal{A}_q(n)=\bigoplus_{s\ge0}\mathcal{A}_q(n)^{(s)}$, and $\mathcal{A}_q(n,\mathbf{1})=\bigoplus_{s=0}^N\mathcal{A}_q(n,\mathbf{1})^{(s)}$ when $\mathbf{char}(q)=l$ are $\mathbb{Z}_+$-graded algebras.

Proposition. *The subspace $\mathcal{A}_q^{(s)}$ of homogeneous elements of degree s is a $U_q(\mathfrak{sl}_n)$-submodule of the quantum divided power (restricted) algebra $\mathcal{A}_q$.*

(1) *If $\mathbf{char}(q)=0$, $\mathcal{A}_q(n)^{(s)}$ is generated by the highest weight vector $x^{(s\,\varepsilon_1)}$ (where $s\,\varepsilon_1=(s,0,\cdots,0)$), which is isomorphic to the simple module $V(s\,\lambda_1)$ (where $\lambda_1=\varepsilon_1$ is the 1-st fundamental weight of $\mathfrak{g}$).*

(2) *If $\mathbf{char}(q)=l\ge3$, $\mathcal{A}_q(n,\mathbf{1})^{(s)}$ is generated by the highest weight vector $x^{(\,(l-1)\varepsilon_1+\cdots+(l-1)\varepsilon_{i-1}+s_i\varepsilon_i)}$ (where $s=(i-1)(l-1)+s_i$, $0\le s_i\le l-1$ for $1\le i\le n$), which is isomorphic to the simple module $V(\lambda)$ (where $\lambda=(l-1-s_i)\lambda_{i-1}+s_i\,\lambda_i$, $\lambda_i=\varepsilon_1+\cdots+\varepsilon_i$ $(1\le i<n)$ is the i-th fundamental weight of $\mathfrak{g}$ with $\lambda_0=\lambda_n=0$).*

Proof. Obviously, we see from formulae **4.1** (iii)–(v) that the action of $U_q(\mathfrak{sl}_n)$ on $\mathcal{A}_q$ stabilizes $\mathcal{A}_q^{(s)}$. Moreover, in the case when $\mathbf{char}(q)=0$, we note that for $1\le i<n$,

$$
\begin{aligned}
e_i(x^{(s\,\varepsilon_1)}) &= 0,\\
\mathcal{K}_i(x^{(s\,\varepsilon_1)}) &= q^{\delta_{i,1}s}\,x^{(s\,\varepsilon_1)}\\
&= q^{\langle s\,\lambda_1,\varepsilon_i-\varepsilon_{i+1}\rangle}\,x^{(s\,\varepsilon_1)},
\end{aligned}
$$

which imply the vector $x^{(s\,\varepsilon_1)}$ is a highest weight vector with highest weight $s\,\lambda_1$. Again for any $x^{(\beta)}\in\mathcal{A}_q(n)^{(s)}$, $|\beta|=s$, set $s_i=s-\sum_{j\le i}\beta_j$ $(1\le i<n)$, that is, $\beta_1=s-s_1$, $\beta_2=s_1-s_2,\cdots,\beta_{n-1}=s_{n-2}-s_{n-1}$, $\beta_n=s_{n-1}$, from formula **4.1** (v) and by induction on s_i, we get

$$
\begin{aligned}
&f_{n-1}^{s_{n-1}}\cdots f_1^{s_1}(x^{(s\,\varepsilon_1)})\\
&\qquad=[s_1]!\cdots[s_{n-1}]!\,x^{((s-s_1)\varepsilon_1+(s_1-s_2)\varepsilon_2+\cdots+(s_{n-2}-s_{n-1})\varepsilon_{n-1}+s_{n-1}\,\varepsilon_n)}\\
&\qquad=[s_1]!\cdots[s_{n-1}]!\,x^{(\beta)},
\end{aligned}
$$

where $[m]! = [m]\cdots[2]\cdot[1]$, for $m \in \mathbb{Z}_+$. Thus we show that $\mathcal{A}_q(n)^{(s)} \cong V(s\,\lambda_1)$ is indeed a simple highest weight module with $s\,\lambda_1$ as its highest weight, where $x^{(s\,\varepsilon_1)}$ is its a highest weight vector.

In the case when $\mathbf{char}(q) = l\,(\geq 3)$, for $0 \leq s \leq N$, there exist i and s_i such that $1 \leq i \leq n$, $s = (i-1)(l-1) + s_i$ with $0 \leq s_i \leq l-1$. Consider the vector $x^{(\mathbf{s})} \in \mathcal{A}_q(n, \mathbf{1})^{(s)}$ where $\mathbf{s} = (l-1)\varepsilon_1 + \cdots + (l-1)\varepsilon_{i-1} + s_i\varepsilon_i$. From formulae **4.1** (iii)–(v) and noting $[\,l\,] = 0$, we conclude that for $1 \leq j < n$,

$$
\begin{aligned}
\mathcal{K}_j(x^{(\mathbf{s})}) &= q^{\langle (l-1-s_i)\lambda_{i-1} + s_i\lambda_i,\, \varepsilon_j - \varepsilon_{j+1}\rangle}\, x^{(\mathbf{s})},\\
e_j(x^{(\mathbf{s})}) &= 0,
\end{aligned}
$$

which imply $x^{(\mathbf{s})}$ is a highest weight vector with highest weight $(l-1-s_i)\lambda_{i-1}+s_i\lambda_i$.

On the other hand, for any $x^{(\beta)} \in \mathcal{A}_q(n,\mathbf{1})^{(s)}$, we have $|\beta\,| = s$, $0 \leq \beta_i \leq l-1$ $(1 \leq i \leq n)$. We denote r by the last ordinal dumber with $\beta_r \neq 0$ for n-tuple $\beta = (\beta_1, \cdots, \beta_n)$. Then $r \geq i$ if $s_i \neq 0$, and $r \geq i-1$ if $s_i = 0$. Hence, in terms of formula **4.1** (v), we obtain that

Case (i): if $s_i \geq \beta_r\,(>0)$, then

$$
\begin{aligned}
f_{r-1}^{\beta_r}\cdots f_i^{\beta_r}(x^{(\mathbf{s})}) &= (\,[\beta_r]!\,)^{r-i}\, x^{(\mathbf{s}-\beta_r\varepsilon_i+\beta_r\varepsilon_r)}\\
&= (\,[\beta_r]!\,)^{r-i}\, x^{(\mathbf{s}')}\, x^{(\beta_r\varepsilon_r)} \neq 0,
\end{aligned}
$$

where $\mathbf{s}' := \mathbf{s} - \beta_r\varepsilon_i = (l-1)\varepsilon_1 + \cdots + (l-1)\varepsilon_{i-1}$.

Case (ii): if $\beta_r > s_i\,(\geq 0)$, then

$$
\begin{aligned}
&f_{r-1}^{\beta_r-s_i}\, f_{r-2}^{\beta_r-s_i}\cdots f_{i-1}^{\beta_r-s_i}\, f_{r-1}^{s_i}\cdots f_i^{s_i}\,(x^{(\mathbf{s})})\\
&\quad= (\,[s_i]!\,[\beta_r-s_i]!\,)^{r-i}\,[s_i+1][s_i+2]\cdots[\beta_r]\, x^{(\mathbf{s}-(\beta_r-s_i)\varepsilon_{i-1}-s_i\varepsilon_i+\beta_r\varepsilon_r)}\\
&\quad= (\,[s_i]!\,[\beta_r-s_i]!\,)^{r-i}\,[s_i+1][s_i+2]\cdots[\beta_r]\, x^{(\mathbf{s}')}\, x^{(\beta_r\varepsilon_r)} \neq 0,
\end{aligned}
$$

where $\mathbf{s}' := \mathbf{s} - (\beta_r - \varepsilon_i)\varepsilon_{i-1} - s_i\varepsilon_i = (l-1)\varepsilon_1 + \cdots ((l-1)\varepsilon_{i-2} + (l-1-\beta_r+s_i)\varepsilon_{i-1}$.

Set $\beta' := \beta - \beta_r\varepsilon_r$ and use an induction on $\mathbf{s}$. At first, the argument holds for $\mathbf{s} = \varepsilon_1 = \lambda_1$ (see the following Remark). Assume that there exists a word ω in $U_q(\mathfrak{sl}_n)$ constructed by some suitable f_j's (where $j < r$) such that $\omega\,(x^{(\mathbf{s}')}) = c\,x^{(\beta')}$ $(c \in k^*)$. Note that $f_j\,(x^{(\beta_r\varepsilon_r)}) = 0$ for those f_j $(j < r)$. Thus we get

$$
\begin{aligned}
\omega\,(x^{(\mathbf{s}')}\, x^{(\beta_r\varepsilon_r)}) &= \omega\,(x^{(\mathbf{s}')})\, x^{(\beta_r\varepsilon_r)}\\
&= c\,x^{(\beta')}\, x^{(\beta_r\varepsilon_r)} = c\,x^{(\beta)} \neq 0.
\end{aligned}
$$

Since $x^{(\beta)}$ (with $|\beta| = s$) is arbitrary, $\mathcal{A}_q(n,\mathbf{1})^{(\mathbf{s})}$ is an indecomposable module generated by the highest weight vector $x^{(\mathbf{s})}$.

Finally, by virtue of **4.1** (v), we find both e_i's and f_i's act nilpotently on $\mathcal{A}_q(n,\mathbf{1})$, namely, $e_i^l|_{\mathcal{A}_q(n,\mathbf{1})} \equiv 0 \equiv f_i^l|_{\mathcal{A}_q(n,\mathbf{1})}$. This implies the $U_q(\mathfrak{sl}_n)$-module $\mathcal{A}_q(n,\mathbf{1})$ is completely reductive. Consequently, we derive that $\mathcal{A}_q(n,\mathbf{1})^{(s)} \cong V((l-1-s_i)\lambda_{i-1} + s_i\lambda_i)$ is a simple highest weight module. $\Box$

Remark. Particularly, $\langle x_1,\cdots,x_n\rangle \cong V(\lambda_1)$ with $e_i(x_j)=\delta_{i+1,j}\,x_i$, $f_i(x_j)=\delta_{ij}\,x_{i+1}$ and $\mathcal{K}_i(x_1)=q^{\langle\lambda_1,\varepsilon_i-\varepsilon_{i+1}\rangle}\,x_1$. Observe that the conclusion (1) of Proposition 4.2 is valid for the quantum n-space $k[A_q^{n|0}]$ when $\mathbf{char}(q)=0$ (see Remark 4.1).

4.3 As the dual object of the quantum n-space (cf. [11]), we can consider its submodule structures of the quantum exterior algebra $k[A_q^{0|n}]=\Lambda_q(n)=\bigoplus_{s=0}^n\Lambda_q(n)_{(s)}$ where $\Lambda_q(n)_{(s)}:=\langle\,x_{i_1}\cdots x_{i_s}\mid 1\le i_1<\cdots<i_s\le n\,\rangle$. The known fact below is independent of $\mathbf{char}(q)$.

Proposition. *The subspace $\Lambda_q(n)_{(s)}$ of homogeneous elements of degree s is a $U_q(\mathfrak{sl}_n)$-submodule of $\Lambda_q(n)$. It is generated by the highest weight vector $x_1\cdots x_s$ and is isomorphic to the simple module $V(\lambda_s)$ (where $\lambda_s=\varepsilon_1+\cdots+\varepsilon_s$ is the s-th fundamental weight of $\mathfrak{sl}_n$). In other words, the quantum exterior algebra $\Lambda_q(n)$ is the direct sum of those all basic simple modules of $U_q(\mathfrak{sl}_n)$.*

Proof. We can identify elements in $\Lambda_q(n)_{(s)}$ with those in tensor algebra $\mathcal{T}(V)$ with a $U_q(\mathfrak{sl}_n)$-action induced from the action on V via the s-th comultiplication $\Delta^{(s)}$, where $V=\langle x_1,\cdots,x_n\rangle$ is the first basic module (see Remark 4.2). Noting $x_i^2=0$ in $\Lambda_q(n)$, we readily obtain that for $1\le i<n$,

$$\begin{aligned}
\mathcal{K}_i(x_1\cdots x_s)&=q^{\delta_{is}}\ x_1\cdots x_s\\
&=q^{\langle\lambda_s,\varepsilon_i-\varepsilon_{i+1}\rangle}\ x_1\cdots x_s,\\
e_i(x_1\cdots x_s)&=0\qquad (\textit{since}\quad x_i^2=0),\\
f_{j_s-1}\cdots f_s(x_1\cdots x_s)&=x_1\cdots x_{s-1}x_{j_s}\qquad (j_s\ge s),
\end{aligned}$$

thus for any $1\le j_1<\cdots<j_s\le n$, one has

$$(f_{j_1-1}\cdots f_1)\cdots(f_{j_s-1}\cdots f_s)\,(x_1\cdots x_s)=x_{j_1}\cdots x_{j_s}.$$

Consequently, $x_1\cdots x_s$ is a highest weight vector of weight λ_s and generates the simple submodule $\Lambda_q(n)_{(s)}$. $\Box$

4.4 Theorem 4.1 has indicated the presentation in $\mathcal{W}_q(2n)$ of the generators of $U_q(\mathfrak{sl}_n)$. Actually, we can describe explicitly the presentation of all "*roots vectors*" of $U_q(\mathfrak{sl}_n)$ under our realization, which is coincident with one of four kinds of roots vectors introduced by G. Lusztig in terms of its braid (automorphism) group action (cf. Lemma 39.3.2, Corollary 40.2.2 in [9]). To do so, we need some notions.

Recall some known facts about $U_q(\mathfrak{sl}_n)$. Let $\alpha_i=\varepsilon_i-\varepsilon_{i+1}$ $(1\le i<n)$ be the simple roots of $\mathfrak{sl}_n$ (see section **1.4**), s_i $(1\le i<n)$ the reflections determined by those α_i respectively and generate the Weyl group $W=S_n$ (the permutation group). The general positive roots in Δ^+ take the form $\alpha_{ij}=\alpha_i+\alpha_{i+1}+\cdots+\alpha_{j-1}=\varepsilon_i-\varepsilon_j$ $(1\le i<j\le n)$ with $\alpha_{i,i+1}=\alpha_i$. Fix a reduced element $\omega_0=s_{i_1}s_{i_2}\cdots s_{i_N}$ of maximal length in W. Then $\alpha=s_{i_1}s_{i_2}\cdots s_{i_{p-1}}(\alpha_{i_p})$ as $p=1,2,\cdots,N$ runs through all the positive roots of Δ^+

Let T_i be the braid automorphism of $U_q(\mathfrak{sl}_n)$ corresponding to s_i $(1\le i<n)$,

which have been introduced as $T''_{i,-1}$ by Lusztig in §37.1.3 [9] and take the form

$$
\begin{aligned}
&T_i(\mathcal{K}_\mu)=\mathcal{K}_{s_i(\mu)}, \qquad T_i(e_i)=-f_i\,\mathcal{K}_i^{-1}, \qquad T_i(f_i)=-\mathcal{K}_i\,e_i;\\
&T_i(e_j)=e_j, \qquad T_i(f_j)=f_j, \qquad (|i-j|>1);\\
&T_i(e_j)=e_i\,e_j-q\,e_j\,e_i, \qquad (|i-j|=1);\\
&T_i(f_j)=f_jf_i-q^{-1}\,f_i\,f_j, \qquad (|i-j|=1)
\end{aligned} \tag{1}
$$

where e_i $(1\le i<n)$ are the simple roots vectors of $U_q(\mathfrak{sl}_n)$ corresponding to the simple roots α_i respectively, and f_i $(1\le i<n)$ the negative simple roots vectors to those $-\alpha_i$. As we know, the elements below

$$e_\alpha=T_{i_1}\cdots T_{i_{p-1}}(e_{i_p}), \qquad f_\alpha=T_{i_1}\cdots T_{i_{p-1}}(f_{i_p})$$

are called roots vectors of $U_q(\mathfrak{sl}_n)$ associated to boots $\pm\alpha=\pm s_{i_1}s_{i_2}\cdots s_{i_{p-1}}(\alpha_{i_p})$ $(p=1,2,\cdots,N)$ respectively, where $\omega_0=s_{i_1}s_{i_2}\cdots s_{i_N}$ is as above. Normally, for the Weyl group of $\mathfrak{sl}_n$, we can take such a reduced element ω_0 as

$$\omega_0=s_1s_2s_1s_3s_2s_1\cdots s_{n-2}s_{n-3}\cdots s_2s_1s_{n-1}s_{n-2}\cdots s_2s_1, \tag{2}$$

which gives rise to a well-known ordering of Δ^+ as

$$\alpha_{12},\ \alpha_{13},\ \alpha_{23},\ \alpha_{14},\ \alpha_{24},\ \alpha_{34},\ \cdots,\ \alpha_{1n},\ \alpha_{2n},\ \cdots,\ \alpha_{n-1,n}. \tag{3}$$

Hence, all the positive roots vectors of $U_q(\mathfrak{sl}_n)$ associated to the ordering (3) of Δ^+ are as follows.

$$
\begin{gathered}
e_{\alpha_{12}}=e_1,\\
e_{\alpha_{13}}=T_1(e_2), \qquad e_{\alpha_{23}}=T_1T_2(e_1)=e_2,\\
e_{\alpha_{14}}=T_1T_2T_1(e_3), \qquad e_{\alpha_{24}}=T_1T_2T_1T_3(e_2), \qquad e_{\alpha_{34}}=T_1T_2T_1T_3T_2(e_1)=e_3,\\
\cdots\cdots\\
e_{\alpha_{1n}}=T_1T_2T_1T_3T_2T_1\cdots T_{n-2}T_{n-3}\cdots T_2T_1(e_{n-1}),\\
e_{\alpha_{2n}}=T_1T_2T_1T_3T_2T_1\cdots T_{n-2}T_{n-3}\cdots T_2T_1T_{n-1}(e_{n-2}),\\
\cdots\cdots\\
e_{\alpha_{n-1,n}}=T_1T_2T_1T_3T_2T_1\cdots T_{n-2}T_{n-3}\cdots T_2T_1T_{n-1}T_{n-2}\cdots T_2(e_1)=e_{n-1}.
\end{gathered} \tag{4}
$$

4.5 We introduce here some q-differential operators in $\mathcal{W}_q(2n)$. Set $E_{ij}:=x_i\partial_j$, for any $1\le i,\ j\le n$. Denote $e_{ij}:=E_{ij}\,\sigma_i$ for $1\le i<j\le n$ and $e_{ij}:=\sigma_j^{-1}\,E_{ij}$ for $1\le j<i\le n$. Since

$$
\begin{aligned}
E_{ij}(x^{(\beta)})&=\big(\,x_i\partial_j\,\big)\,(x^{(\beta)})\\
&=[\beta_i+1]\,q^{\varepsilon_i*(\beta-\varepsilon_j)-\varepsilon_j*\beta}\,x^{(\beta+\varepsilon_i-\varepsilon_j)}\\
&=[\beta_i+1]\,q^{(\varepsilon_i-\varepsilon_j)*\beta-\varepsilon_i*\varepsilon_j}\,x^{(\beta+\varepsilon_i-\varepsilon_j)}\\
&=[\beta_i+1]\,q^{-\sum_{i\le s<j}\beta_s}\,x^{(\beta+\varepsilon_i-\varepsilon_j)}, \qquad (i<j)\\
E_{ij}(x^{(\beta)})&=\big(\,x_i\partial_j\,\big)\,(x^{(\beta)})\\
&=[\beta_i+1]\,q^{(\varepsilon_i-\varepsilon_j)*\beta-\varepsilon_i*\varepsilon_j}\,x^{(\beta-\varepsilon_j+\varepsilon_i)}\\
&=[\beta_i+1]\,q^{\sum_{j\le s<i}\beta_s-1}\,x^{(\beta-\varepsilon_j+\varepsilon_i)}, \qquad (i>j)
\end{aligned}
$$

thus we get

$$e_{ij}(x^{(\beta)})=(E_{ij}\sigma_i)\,(x^{(\beta)})=[\beta_i+1]\,q^{-\sum_{i<s<j}\beta_s}\,x^{(\beta+\varepsilon_i-\varepsilon_j)},\quad (i<j) \tag{1}$$

$$e_{ij}(x^{(\beta)})=(\sigma_j^{-1}E_{ij})\,(x^{(\beta)})=[\beta_i+1]\,q^{\sum_{i>s>j}\beta_s}\,x^{(\beta-\varepsilon_j+\varepsilon_i)}.\quad (i>j) \tag{2}$$

Lemma. (i) *If $i<j$, then for any $i<k<j$, we have $e_{ij}=e_{ik}\,e_{kj}-q\,e_{kj}\,e_{ik}$;*

(ii) *If $i>j$, then for any $i>k>j$, we have $e_{ij}=e_{ik}\,e_{kj}-q^{-1}\,e_{kj}\,e_{ik}$;*

(iii) *For $i<j$, we have $e_{ij}\,e_{ji}-e_{ji}\,e_{ij}=\frac{\sigma_i\,\sigma_j^{-1}-\sigma_i^{-1}\,\sigma_j}{q-q^{-1}}$, where $\sigma_i\,\sigma_j^{-1}=\mathcal{K}_{\varepsilon_i-\varepsilon_j}$.*

Proof. (i) For $i<k<j$. Since $[m+1]-q\,[m]=q^{-m}$, it follows from (1) that

$$\begin{aligned}
&\big(\,e_{ik}\,e_{kj}-q\,e_{kj}\,e_{ik}\,\big)\,(x^{(\beta)})\\
&\quad=[\beta_k+1]\,q^{-\sum_{k<s<j}\beta_s}\,e_{ik}(x^{(\beta+\varepsilon_k-\varepsilon_j)})-q\,[\beta_i+1]\,q^{-\sum_{i<s<k}\beta_s}\,e_{kj}(x^{(\beta+\varepsilon_i-\varepsilon_k)})\\
&\quad=(\,[\beta_k+1]\,[\beta_i+1]-q\,[\beta_i+1]\,[\beta_k]\,)\,q^{-\sum_{i<s<j}\beta_s+\beta_k}\,x^{(\beta+\varepsilon_i-\varepsilon_j)}\\
&\quad=[\beta_i+1]\,q^{-\sum_{i<s<j}\beta_s}\,x^{(\beta+\varepsilon_i-\varepsilon_j)}=e_{ij}\,(x^{(\beta)}),\qquad \forall\,x^{(\beta)}\in\mathcal{A}_q.
\end{aligned}$$

(ii) For $i>k>j$. Since $[m+1]-q^{-1}\,[m]=q^m$, it follows from (2) that

$$\begin{aligned}
&\big(\,e_{ik}\,e_{kj}-q^{-1}\,e_{kj}\,e_{ik}\,\big)\,(x^{(\beta)})\\
&\quad=[\beta_k+1]\,q^{\sum_{k>s>j}\beta_s}\,e_{ik}(x^{(\beta+\varepsilon_k-\varepsilon_j)})-q^{-1}\,[\beta_i+1]\,q^{\sum_{i>s>k}\beta_s}\,e_{kj}(x^{(\beta+\varepsilon_i-\varepsilon_k)})\\
&\quad=(\,[\beta_k+1]\,[\beta_i+1]-q^{-1}\,[\beta_i+1]\,[\beta_k]\,)\,q^{\sum_{i>s>j}\beta_s-\beta_k}\,x^{(\beta+\varepsilon_i-\varepsilon_j)}\\
&\quad=[\beta_i+1]\,q^{\sum_{i>s>j}\beta_s}\,x^{(\beta+\varepsilon_i-\varepsilon_j)}=e_{ij}\,(x^{(\beta)}),\qquad \forall\,x^{(\beta)}\in\mathcal{A}_q.
\end{aligned}$$

(iii) For $i<j$, by (1) & (2), we get

$$\begin{aligned}
&\big(\,e_{ij}\,e_{ji}-e_{ji}\,e_{ij}\,\big)\,(x^{(\beta)})\\
&\quad=(\,[\beta_i]\,[\beta_j+1]-[\beta_j]\,[\beta_i+1]\,)\,x^{(\beta)}\\
&\quad=[\beta_i-\beta_j]\,x^{(\beta)}\\
&\quad=\frac{\sigma_i\,\sigma_j^{-1}-\sigma_i^{-1}\,\sigma_j}{q-q^{-1}}\,(x^{(\beta)}),\qquad \forall\,x^{(\beta)}\in\mathcal{A}_q.
\end{aligned}$$

Thus we conclude the proof. □

We introduce two q-brackets as follows.

$$[e_{ik},e_{kj}]_q:=e_{ik}\,e_{kj}-q\,e_{kj}\,e_{ik},\qquad \textit{for}\quad i<k<j. \tag{3}$$

$$[e_{ik},e_{kj}]_{q^{-1}}:=e_{ik}\,e_{kj}-q^{-1}\,e_{kj}\,e_{ik}.\qquad \textit{for}\quad i>k>j. \tag{4}$$

So we arrive at $e_{ij}=[e_{ik},e_{kj}]_q$ for $i<j$, and $e_{ij}=[e_{ik},e_{kj}]_{q^{-1}}$ for $i>j$.

On the other hand, the preceding Lemma also indicates e_{ij} expressed by the q-brackets in two cases are independent of the choice of k. Hence, we deduce that

$$\begin{aligned}
e_{ij}&=[\cdots[e_{i,i+1},e_{i+1,i+2}]_q,\cdots,e_{j-2,j-1}]_q,e_{j-1,j}]_q\\
&=[e_{i,i+1},[e_{i+1,i+2},\cdots,[e_{j-2,j-1},e_{j-1,j}]_q\cdots]_q,\qquad (i<j)\\
e_{ij}&=[\cdots[e_{i,i-1},e_{i-1,i-2}]_{q^{-1}},\cdots,e_{j+2,j+1}]_{q^{-1}},e_{j+1,j}]_{q^{-1}}\\
&=[e_{i,i-1},[e_{i-1,i-2},\cdots,[e_{j+2,j+1},e_{j+1,j}]_{q^{-1}}\cdots]_{q^{-1}},\qquad (i>j)
\end{aligned} \tag{5}$$

Remark. It should be mentioned that the validity of q-bracket $[\,,\,]_q$ defined in (3) (resp. $[\,,\,]_{q^{-1}}$ in (4)) only involves one direction of an ordering of e_{ij}, which is a different point from the defining relation of T_i on e_j in **4.4** (1), however, it doesn't affect our conclusion of the next Proposition 4.6.

4.6 We now shall show that the positive roots vectors $e_{\alpha_{ij}}$ of $U_q(\mathfrak{sl}_n)$ given in **4.4** (4) just correspond to those q-differential operators e_{ij} described in **4.5** (1) in the sense of our realization (cf. Theorem 4.1), that is, if we identify e_i (resp. f_i) with $e_{i,i+1}$ (resp. $e_{i+1,i}$), as well as $\mathcal{K}_i$ with $\sigma_i\,\sigma_{i+1}^{-1}$ for $1\le i<n$, then all positive roots vectors $e_{\alpha_{ij}}$ can be identified with the above q-differential operators e_{ij}. More generally, we have

Proposition. *Identifying the generators of $U_q(\mathfrak{sl}_n)$ with the certain q-differential operators in $\mathcal{W}_q(2n)$, i.e. $e_i := e_{i,i+1}$, $f_i := e_{i+1,i}$, $\mathcal{K}_i := \sigma_i\,\sigma_{i+1}^{-1}$ with $1\le i<n$. Then we have*

(i) *e_{ij} $(i<j)$ correspond to the positive root vectors $e_{\alpha_{ij}}$ associated to those positive roots $\alpha_{ij}=\varepsilon_i-\varepsilon_j$ $(i<j)$, i.e. $e_{\alpha_{ij}}:=e_{ij}$.*

(ii) *e_{ij} $(i>j)$ correspond to the negative root vectors $f_{\alpha_{ji}}$ associated to the negative roots $-\alpha_{ji}=\varepsilon_i-\varepsilon_j$ $(i>j)$, i.e. $f_{\alpha_{ji}}:=e_{ij}$.* □

Obviously, it suffices to prove the first claim. For this purpose, according to the ordering of Δ^+ made in **4.4** (3), we will use an induction on the length of ordered subword of ω_0 as in **4.4** (2) to show the required identification relation of all positive roots vectors $e_{\alpha_{ij}}$ $(i<j)$.

To do so, we will first establish an auxiliary Lemma and make use of the following facts due to Lusztig (cf. §39.2.4 in [9]) in our argument.

$$T_iT_jT_i = T_jT_iT_j,\qquad |i-j|=1, \tag{1}$$

$$T_iT_j(e_i)=e_j,\qquad |i-j|=1, \tag{2}$$

$$T_i(e_j)=e_j,\qquad |i-j|>1. \tag{3}$$

Lemma. *With the identification as the preceding Proposition, we have*

(i) *If $i+1<j$, then $[e_{ij},T_i(e_i)]_q = q\,f_i\,\mathcal{K}_i^{-1}\,e_{ij}-e_{ij}\,f_i\,\mathcal{K}_i^{-1}=e_{i+1,j}$.*

(ii) *If $i+1<j$, then $[T_i(f_i),e_{ji}]_{q^{-1}} = q^{-1}\,e_{ji}\,\mathcal{K}_ie_i-\mathcal{K}_ie_i\,e_{ji}=e_{j,i+1}$.*

Proof. (i) When $i+1<j$, $\forall\, x^{(\beta)}\in\mathcal{A}_q$, using **4.4** (1), **4.5** (1) & (2), we have

$$\begin{aligned}
&\big(\,q\,f_i\,\mathcal{K}_i^{-1}\,e_{ij}-e_{ij}\,f_i\,\mathcal{K}_i^{-1}\,\big)(x^{(\beta)})\\
&\qquad = q^{1-\sum_{i<s<j}\beta_s}[\beta_i+1]\,f_i\,\mathcal{K}_i^{-1}(x^{(\beta+\varepsilon_i-\varepsilon_j)})\\
&\qquad\quad - q^{-\beta_i+\beta_{i+1}}\,[\beta_{i+1}+1]\,e_{ij}(x^{(\beta-\varepsilon_i+\varepsilon_{i+1})})\\
&\qquad = q^{-\sum_{i\le s<j}\beta_s+\beta_{i+1}}\,[\beta_i+1]\,[\beta_{i+1}+1]\,x^{(\beta+\varepsilon_{i+1}-\varepsilon_j)}\\
&\qquad\quad - q^{-\beta_i+\beta_{i+1}-\sum_{i<s<j}\beta_s-1}\,[\beta_{i+1}+1]\,[\beta_i]\,x^{(\beta+\varepsilon_{i+1}-\varepsilon_j)}\\
&\qquad = q^{-\beta_i-\sum_{i+1<s<j}\beta_s}\,[\beta_{i+1}+1]\,(\,[\beta_i+1]-q^{-1}\,[\beta_i]\,)\,x^{(\beta+\varepsilon_{i+1}-\varepsilon_j)}\\
&\qquad = q^{-\sum_{i+1<s<j}\beta_s}\,[\beta_{i+1}+1]\,x^{(\beta+\varepsilon_{i+1}-\varepsilon_j)} = e_{i+1,j}(x^{(\beta)}),
\end{aligned}$$

so the conclusion is true.

(ii) When $j > i+1$, $\forall\, x^{(\beta)} \in \mathcal{A}_q$, using **4.4** (1), **4.5** (1) & (2), we have

$$\begin{aligned}
&\big(\, q^{-1}\, e_{ji}\, \mathcal{K}_i\, e_i - \mathcal{K}_i\, e_i\, e_{ji} \,\big)(x^{(\beta)})\\
&\qquad\qquad\qquad = \big(\, q^{-1+\beta_i-\beta_{i+1}+1+\sum_{j>s>i}\beta_s}\, [\beta_i+1]\, [\beta_j+1]\\
&\qquad\qquad\qquad\quad - q^{\sum_{j>s>i}\beta_s+\beta_i-\beta_{i+1}+1}\, [\beta_i]\, [\beta_j+1]\,\big)\, x^{(\beta+\varepsilon_j-\varepsilon_{i+1})}\\
&\qquad\qquad\qquad = q^{\sum_{j>s>i+1}\beta_s}\, [\beta_j+1]\, x^{(\beta+\varepsilon_j-\varepsilon_{i+1})} = e_{j,i+1}(x^{(\beta)}),
\end{aligned}$$

so the claim is true. $\square$

We are now in the position to show Proposition 4.6.

Proof of Proposition 4.6. We suffice to verify our identification $e_{\alpha_{ij}} := e_{ij}$ for the case when $i < j$. We adopt the ordering of positive roots vectors $e_{\alpha_{ij}}$ as in the list **4.4** (4) and use an induction on k where $1 \le k < n$.

For $k = 1$, this is just the identification $e_{\alpha_1} = e_1 := e_{12}$.

For $k = 2$, by the identification and using **4.5** (3) & **4.6** (2), we have

$$\begin{aligned}
e_{\alpha_{13}} &= T_1(e_2) = e_1\, e_2 - q\, e_2\, e_1 := e_{12}\, e_{23} - q\, e_{23}\, e_{12}\\
&= [e_{12}, e_{23}]_q = e_{13},\\
e_{\alpha_{23}} &= T_1T_2(e_1) = e_2 := e_{23}.
\end{aligned}$$

Assume that the claim is valid for the case $U_q(\mathfrak{sl}_{n-1})$. Thus for the case $U_q(\mathfrak{sl}_n)$, we further need to show that in the list **4.4** (4) there hold

$$e_{\alpha_{1n}} := e_{1n}, \quad e_{\alpha_{2n}} := e_{2n}, \ \cdots, \quad e_{\alpha_{n-1,n}} := e_{n-1,n}.$$

At first, according to the above assumption, we have

$$e_{\alpha_{1,n-1}} = T_1T_2T_1T_3T_2T_1\cdots T_{n-4}\cdots T_2T_1T_{n-3}\cdots T_2T_1(e_{n-2}) := e_{1,n-1}.$$

Thus, noting (3) and using **4.5** (3), we have

$$\begin{aligned}
e_{\alpha_{1n}} &= T_1T_2T_1T_3T_2T_1\cdots T_2T_1T_{n-3}\cdots T_2T_1T_{n-2}\cdots T_2T_1(e_{n-1})\\
&= T_1T_2T_1T_3T_2T_1\cdots T_2T_1T_{n-3}\cdots T_2T_1T_{n-2}(e_{n-1})\\
&= T_1T_2T_1T_3T_2T_1\cdots T_{n-3}\cdots T_2T_1\,(e_{n-2}\, e_{n-1} - q\, e_{n-1}\, e_{n-2}\,)\\
&= T_1T_2T_1T_3T_2T_1\cdots T_{n-3}\cdots T_2T_1\,(e_{n-2})\cdot e_{n-1}\\
&\qquad - q\, e_{n-1}\cdot T_1T_2T_1T_3T_2T_1\cdots T_2T_1T_{n-3}\cdots T_2T_1\,(e_{n-2})\\
&:= e_{1,n-1}\, e_{n-1,n} - q\, e_{n-1,n}\, e_{1,n-1}\\
&= [e_{1,n-1},\, e_{n-1,n}]_q\\
&= e_{1n}.
\end{aligned}$$

Let us further suppose that we have proved the identification

$$e_{\alpha_{jn}} = T_1T_2T_1T_3T_2T_1\cdots T_{n-2}T_{n-3}\cdots T_2T_1T_{n-1}T_{n-2}\cdots T_{n-j+1}(e_{n-j}) := e_{jn},$$

for $1 \leq j < n-1$. Now we want to show $e_{\alpha_{j+1,n}} := e_{j+1,n}$. Note that

$$\begin{aligned}
T_1T_2T_1T_3T_2T_1\cdots T_j\cdots T_2(f_1) &= f_j,\\
T_1T_2T_1T_3T_2T_1\cdots T_j\cdots T_2(\mathcal{K}_1) &= \mathcal{K}_j,\\
T_1T_2T_1T_3T_2T_1\cdots T_j\cdots T_2T_1(e_1) &= T_1T_2T_1T_3T_2T_1\cdots T_j\cdots T_2(-f_1\,\mathcal{K}_1^{-1})\\
&= -f_j\,\mathcal{K}_j^{-1} = T_j(e_j).
\end{aligned}$$

Using **4.6** (2) and Lemma 4.6 (i), we get

$$\begin{aligned}
e_{\alpha_{j+1,n}} &= T_1T_2T_1T_3T_2T_1\cdots T_{n-2}T_{n-3}\cdots T_2T_1T_{n-1}T_{n-2}\cdots T_{n-j+1}T_{n-j}(e_{n-j-1})\\
&= T_1T_2T_1T_3T_2T_1\cdots T_{n-2}T_{n-3}\cdots T_2T_1T_{n-1}T_{n-2}\cdots T_{n-j+1}(\,[e_{n-j},e_{n-j-1}]_q)\\
&= [\,T_1T_2T_1T_3T_2T_1\cdots T_{n-2}T_{n-3}\cdots T_2T_1T_{n-1}T_{n-2}\cdots T_{n-j+1}(e_{n-j}),\\
&\qquad T_1T_2T_1T_3T_2T_1\cdots T_{n-2}T_{n-3}\cdots T_2T_1T_{n-1}T_{n-2}\cdots T_{n-j+1}(e_{n-j-1})\,]_q\\
&:= [\,e_{jn}, T_1T_2T_1\cdots T_{n-3}\cdots T_2T_1T_{n-2}T_{n-3}\cdots T_{n-j}T_{n-j-1}T_{n-j-2}(e_{n-j-1})\,]_q\\
&= [\,e_{jn}, T_1T_2T_1T_3T_2T_1\cdots T_{n-3}T_{n-4}\cdots T_{n-j-1}T_{n-j-2}T_{n-j-3}(e_{n-j-2})\,]_q\\
&\qquad\quad \cdots\cdots\\
&= [\,e_{jn}, T_1T_2T_1T_3T_2T_1\cdots T_{j+1}T_j\cdots T_2T_1(e_2)\,]_q\\
&= [\,e_{jn}, T_1T_2T_1T_3T_2T_1\cdots T_jT_{j-1}\cdots T_2T_1(e_1)\,]_q\\
&= [\,e_{jn}, T_j(e_j)\,]_q\\
&= e_{j+1,n}.
\end{aligned}$$

Therefore, we complete the proof. □

Acknowledgements

The author would like to express his gratitude to the National Natural Science Foundation of China (Grant No. 19731004), the Science Foundation of the University Doctoral Program CNCE, the Ministry of Education of China and the Shanghai Scientific and Technical Commission, as well as the Institute of Mathematics of ECNU, for their supports. Finally, the author gratefully acknowledges the referee for his/her very helpful comments so that a couple of paragraphs' expositions are made clearer, especially, two gaps originally appeared in Theorem 2.3 and Proposition 2.4 are removed.

References

1. E. Abe, *Hopf Algebras*, Cambridge University Press, 1980.
2. H.H. Andersen, P. Polo & K. Wen, Representations of quantum algebras, Invent. Math. **104** (1990), 1–59.
3. J. Ding & I.B. Frenkel, *Spinor and oscillator representations of quantum groups*, Progress in Mathematics **124** (1994), 127–165.
4. V.G. Drinfeld, *Quantum groups*, in "Proceedings ICM", Berkeley, Amer. Math. Soc. (1987), 798–820.
5. T. Hayashi, *Q-Analogues of Cliford and Weyl algebras — spinor and oscillator representations of quantum enveloping algebras*, Comm. Math. Phys. **127** (1990), 129–144.
6. J.C. Jantzen, *Lectures on Quantum Groups*, Graduate Studies in Mathematics, vol. 6, Amer. Math. Soc., 1996.
7. C. Kassel, *Quantum Groups*, GTM 155, Springer-Verlag, 1995.
8. G. Lusztig, *Modular representations and quantum groups*, Contemp. Math. AMS **82** (1989), 59–77.
9. ———, *Introduction to Quantum Groups*, Progress in Mathematics, vol. 110, Birkhäuser, 1993.
10. S. Majid, $\mathbb{C}$-*statistical quantum groups and Weyl algebras*, J. Math. Phys. **33 (10)** (1992), 3431–3444.
11. Yu.I. Manin, *Quantum Group and Non-Commutative Gemmetry*, Université de Montréal, 1988.
12. S. Montgomery & S.P. Smith, *Skew derivations and* $U_q(sl(2))$, Israel J. of Math. **72(1-2)** (1990), 158–166.
13. B. Parshall & J.P. Wang, *Quantum Linear Groups*, vol. 89 No. 439, Amer. Math. Soc., 1991.
14. N.Yu. Reshetikhin, L.A. Takhtajan & L.D. Faddeev, *Quantization of Lie groups and Lie algebras*, Algebra and Anal. **1** (1989), (Leningrad Math. J. 1 [Engl. transl. 193–225 (1990)]), 178–206.
15. M. Scheunert, *The Theory of Lie Superalgebras, An Introduction*, Lect. Notes. Math. 716, Springer-Verlag, 1979.
16. M. Sweedler, *Hopf Algebras*, Benjamin, New York, 1974.
17. J. Wess & B. Zumino, *Covariant differential calculus on the quantum hyperplane*, Nucl. Phys. B (Proc. Supp.) **18B** (1990), 302–312.

Department of Mathematics, East China Normal University, Shanghai 200062, China

E-mail address: nhhu@math.ecnu.edu.cn